7-2 (p.237)
(a)

$$\begin{aligned} E(\bar{X}) &= E(\frac{1}{2}X_1 + \frac{1}{2}X_2) \\ &= \frac{1}{2}E(X_1) + \frac{1}{2}E(X_2) \\ &= \frac{1}{2}\mu + \frac{1}{2}\mu \\ &= \mu \end{aligned}$$

$$\begin{aligned} E(U) &= E(\frac{1}{3}X_1) + E(\frac{2}{3}X_2) \\ &= \frac{1}{3}E(X_1) + \frac{2}{3}E(X_2) \\ &= \frac{1}{3}\mu + \frac{2}{3}\mu \\ &= \mu \end{aligned}$$

Both $\bar{X}$ and U are unbiased, since the expectation of both of these estimators is equal to the population mean μ.

(b)

$$\begin{aligned} Var(\bar{X}) &= Var(\frac{1}{2}X_1 + \frac{1}{2}X_2) \\ &= \frac{1}{4}Var(X_1) + \frac{1}{4}Var(X_2) \\ &= \frac{1}{2}\sigma^2 \end{aligned}$$

$$\begin{aligned} Var(U) &= Var(\frac{1}{3}X_1 + \frac{2}{3}X_2) \\ &= \frac{1}{9}Var(X_1) + \frac{4}{9}Var(X_2) \\ &= \frac{5}{9}\sigma^2 \end{aligned}$$

Efficiency of $\bar{X}$ compared to U:

$$\frac{Var(U)}{Var(\bar{X})} = \frac{\frac{5}{9}\sigma^2}{\frac{1}{2}\sigma^2} = \frac{10}{(9}$$

We use this simpler efficiency formula since both estimators are unbiased. So, $\bar{X}$ is approximately 11% more efficient than U.

7-6 (p. 238)
(a)

$$\begin{aligned} E(X) &= \Sigma x p(x) \\ &= (8)(\frac{1}{4}) + (10)(\frac{1}{4}) + (11)(\frac{1}{2}) \\ &= 10 \end{aligned}$$

Since true length is 10, so X is unbiased.

(b)

$$\begin{aligned} E(Y) &= \Sigma y p(y) \\ &= (4)(\frac{1}{2}) + (6)(\frac{1}{2}) \\ &= 5 \end{aligned}$$

Since true length is 5, so Y is unbiased.

(c) Remember, that X and Y are independent.

$$\begin{aligned} E(A) &= \Sigma\Sigma xyp(x,y) \\ &= \Sigma\Sigma xyp(x)p(y) \\ &= (8)(4)(\frac{1}{8}) + (8)(6)(\frac{1}{8}) + (10)(4)(\frac{1}{8}) + (10)(6)(\frac{1}{8}) + (11)(4)(\frac{1}{4}) + (11)(6)(\frac{1}{4}) \\ &= 50 \end{aligned}$$

The true area is $5 \times 10 = 50$, so A is unbiased.

7-8 (p. 242)
(a) Gun A (not clamped down hard enough): left-most target
Gun B (clamped down but pointed left): right-most target
Gun C (clamped down correctly): target in the middle

(b) Gun B is biased. Guns B and C appear to have minimum variance (can't easily distinguish if one has smaller variance from the other). Most efficient (lowest MSE) is Gun C. the largest MSE is the least efficient. It's hard to tell whether A or B is least efficient. But if you had to guess:

- It looks like A is unbiased, so MSE(A) = Var(A).
- It looks like the standard deviation of A ($\sqrt{Var(A)}$) may be about equal to the absolute value of the bias of B.
- Then MSE(B) = Var(B) + Var(A)
- Since Var(B) $>$ 0, MSE(B)$>$MSE(A), so B is least efficient.

Thanks to E. Iversen and his TAs from last term for this answer.

7-10 (p. 243)

(a) For μ, the population mean, we use as $p(x)$ the frequencies for the whole population of interest.

$$\begin{aligned}\mu &= E(X)\\ &= \Sigma x p(x)\\ &= (0)(0.40)+(1)(0.24)+(2)(0.20)+(3)(.12)+(4)(0.04)\\ &= 1.16\end{aligned}$$

(b) For survey (i), use as $p(x)$ the frequencies for the subpopulation who would respond.

$$\begin{aligned}E(X_1) &= (0)(0.62)+(1)(0.21)+(2)(0.12)+(3)(.04)+(4)(0.01)\\ &= 0.61\end{aligned}$$

$$\begin{aligned}E(\bar{X}_1) &= \frac{1}{n_1}\Sigma_{i=1} n_1 E(X_1)\\ &= \frac{1}{n_1} n_1(0.61)\\ &= 0.61\end{aligned}$$

So, survey (i) is biased, because the expectation of the sample mean $\bar{X}_1$ is not equal to the population mean of 1.16.

For survey (ii), use as $p(x)$ the frequencies for the whole population (since the non-response bias is not present, the sample is randomly chosen from the whole population of interest). The members of this sample should then follow the same probability distribution as the total population.

$$\begin{aligned}E(\bar{X}_2) &= \frac{1}{n}\Sigma_{i=1} n_2 E(X_2)\\ &= \frac{1}{n_2} n_2 E(X)\\ &= 1.16\end{aligned}$$

So, survey (ii) is not biased, because the expectation of the sampe mean $\bar{X}_2$ is equal to the population mean.

(c)

$$\begin{aligned}MSE &= E[(U-\theta)^2]\\ &= Var(U) + (\text{bias of} U)^2\end{aligned}$$

To find the MSE of U, we need to know the variance of U.

In the case of survey (i), our estimator U is $\bar{X}_1$. We will need to know the variance of the subpopulation who will respond to this survey.

$$\begin{aligned}Var(X_1) &= E(X_1^2)+[E(X_1)]^2\\ &= [(0)(0.62)+(1)(0.21)+(4)(0.12)+(9)(0.04)+(16)(0.01)]-(0.61)^2\\ &= 0.8379\end{aligned}$$

$$\begin{aligned} Var(\bar{X}_1) &= \frac{Var(X_1)}{n_1} \\ &= \frac{0.8379}{200} \end{aligned}$$

$$\begin{aligned} MSE(\bar{X}_1) &= Var(\bar{X}_1) + (\text{bias of} \bar{X}_1)^2 \\ &= \frac{0.8379}{200} + (0.61 - 1.16)^2 \\ &\approx 0.3067 \end{aligned}$$

In the case of survey (ii), our estimator U is $\bar{X}_2$. We will need to know the variance of the total population (since this is a random sample with the same probability distribution as the population of interest).

$$\begin{aligned} Var(X_2) &= E(X_2^2) + [E(X_2)]^2 \\ &= [(0)(0.40) + (1)(0.24) + (4)(0.20) + (9)(0.12) + (16)(0.04)] - (1.16)^2 \\ &= 1.4144 \end{aligned}$$

$$\begin{aligned} Var(\bar{X}_2) &= \frac{Var(X_2)}{n_2} \\ &= \frac{1.4144}{25} \end{aligned}$$

$$\begin{aligned} MSE(\bar{X}_2) &= Var(\bar{X}_2) + (\text{bias of} \bar{X}_2)^2 \\ &= \frac{1.4144}{25} + (1.16 - 1.16)^2 \\ &\approx 0.0566 \end{aligned}$$

So, the smaller sample (25 executives with enough follow-up to get a 100% response rate) has the smaller MSE.