Probability

In Handout 2 we have seen basics of descriptive statistics. Descriptive statistics deal only with the data in hand. The statistician may be interested in drawing conclusions and suggesting decisions about a large set of data on which he has only limited knowledge. That knowledge is obtained by screening a subset of data. Desirable and the most informative subsets are the subsets selected at random. That is how the randomness made it's way into statistics. In order to understand this *inferential* aspect of statistics, an understanding of the concept of probability is essential.

> Probability is a formal, mathematical language for efficient dealing with uncertainties which necessarily arise in inferential statistics.

In this chapter we consider briefly some of the basic ideas of probability. Some of the concepts are illustrated by straightforward and familiar examples.

1 Sample Spaces and Events

We start with the description of some basic terms. Any phenomenon whose outcome is uncertain, such as an outcome of tossing a coin, your grade in STA 110 course, or the weather tomorrow, can be viewed as a **(stochastic) experiment**. The set of *all possible outcomes* of an experiment is called **sample space**, and is denoted by $\mathcal{S}$. An **event** is any *specific set* of outcomes, or a subset of S. We will denote different events by capital letters such as A, B, C, etc.

The gambling, bad habit in real life, provides nice models for describing basic probability. The first probability problems from the time of Pascal, Leibnitz, Fermat, etc. originate in hazard games. Let an experiment consists of rolling a single six-sided die. A natural sample space $\mathcal{S}$ contains six possible outcomes: one dot, two dots, three dots, and so on. We write this symbolically as $\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$.

Suppose we are interested in the event A "six dots." A includes only one of the six possible outcomes in S: A = $\{6\}$. Defining event B as "at least three dots" means B would include four of the six possible outcomes in $\mathcal{S}$: $B = \{3, 4, 5, 6\}$. That is, the event B occurs if *any* of the bracketed outcomes occurs - if 3, 4, 5, or 6 dots show.

2 Probability

The intuition of chance and probability develops at very early ages.[1] However, a formal, precise definition of the probability is elusive.

If the experiment can be repeated potentially infinitely many times, then the probability of an event can be defined through relative frequencies. For instance, if we rolled a die repeatedly, we could construct a frequency distribution table showing how many times each face came up. These frequencies (n_i) can be expressed as proportions or relative frequencies by dividing them by the total number of tosses $n : f_i = n_i/n$. If we saw six dots showing on 107 out of 600 tosses, that face's proportion or relative frequency is $f_6 = 107/600 = 0.178$ As more tosses are made, we "expect" the proportion of sixes to stabilize around $\frac{1}{6}$.

For example, **S** function dice, simulates rolls of a fair die.

```
dice_function(n)
{
        freq.prob.list <- NULL
        six.num <- 0
        for(i in 1:n) {
                if(runif(1) < 1/6)
                        six.num <- six.num + 1
                freq.prob.list <- c(freq.prob.list, six.num/i)
        }
        return(freq.prob.list)
}
```

`dice(n)` will give a sequence of relative frequencies of "6" in rolls $1, 2, \ldots n$.

```
> dice(30)
 [1] 1.0000000 0.5000000 0.6666667 0.7500000 0.6000000 0.5000000 0.4285714
 [8] 0.3750000 0.3333333 0.3000000 0.2727273 0.2500000 0.2307692 0.2142857
[15] 0.2000000 0.1875000 0.1764706 0.1666667 0.1578947 0.1500000 0.1428571
[22] 0.1363636 0.1304348 0.1250000 0.1200000 0.1153846 0.1481481 0.1428571
```

To get the relative frequency of "6" after 10000 trials we take 10000th member of the sequence `dice(10000)`.

```
> dice(10000)[10000]
[1] 0.1652
```

Famous Coin Tosses: Buffon tossed a coin 4040 times. Heads appeared 2048 times. K. Pearson tossed a coin 12000 times and 24000 times. The heads appeared 6019 times and 12012, respectively. For these three tosses the relative frequencies of heads are 0.5049, 0.5016,and 0.5005.

[1] Piaget, J. and Inhelder B. *The Origin of the Idea of Chance in Children*, W. W. Norton & Comp., N.Y.

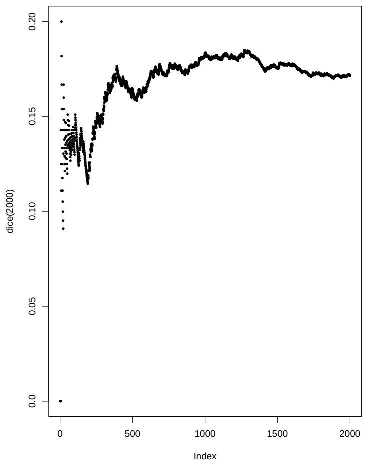

Figure 1: Relative frequency of 6 in 2000 rolls of a fair die

What if the experiments can not be repeated? For example what is probability that guinea pig Squiki survives the treatment by a particular drug. Or "the experiment" of you taking STA 110 course. It is legitimate to ask for a probability of getting an A+ or an F (:-(. In such cases we can define probability **subjectively** as a measure of strength of belief.

The symmetry properties of the experiment lead to the classical definition of probability. An ideal die is symmetric. All sides are "equiprobable". The probability of 6, in our example is a ratio of the number of *favorable* outcomes (in our example only one favorable outcome, namely, 6 itself) and the number of all possible outcomes, 1/6.[2]

(**Frequentist**) An event's **probability** is the proportion of times that we expect the event to occur, if the experiment were repeated a large number of times.

(**Subjectivist**) A subjective **probability** is an individual's degree of belief in the occurrence of an event.

(**Classical**) An event's **probability** is the ratio of the number of favorable outcomes and possible outcomes in the (symmetric) experiment.

[2]This definition is attacked by philosophers because of the fallacy called *circulus vitiosus*. One defines the notion of probability supposing that outcomes are equi**probable**.

Term	*Description*	*Example*
Experiment	Phenomenon where outcomes are uncertain	Single throws of a six-sided die
Sample space	Set of all outcomes of the experiment	$S = \{1,2,3,4,5,6\}$, $(1,2,3,4,5,$ *or* 6 dots show)
Event	A collection of outcomes; a subset of S	$A = \{3\}$ (3 dots show), $B = \{3,4,5,$ *or* $6\}$ (3, 4, 5, or 6 dots show) or 'at least three dots show'
Probability	A number between 0 and 1 assigned to an event.	$P(A) = \frac{1}{6}$. $P(B) = \frac{4}{6}$.

Sure event occurs *every time* an experiment is repeated and has the probability 1. Sure event is in fact the sample space $\mathcal{S}$.

An event that *never* occurs when an experiment is performed is called **impossible event**. The probability of an impossible event, denoted usually by $\emptyset$ is 0.

For any event A, the probability that A will occur is a number between 0 and 1, inclusive:

$$0 \le P(A) \le 1,$$

$$P(\emptyset) = 0, \qquad P(\mathcal{S}) = 1.$$

The **intersection** (product) $A \cdot B$ of two events A and B is an event that occurs if both events A **and** B occur. The key word in the definition of the intersection is **and.**

In the case when the events A and B are independent the probability of the intersection is the product of probabilities: $P(A \cdot B) = P(A)P(B)$.

Example: Two consecutive flipps of a fair coin.

Remark: "Intuitive independence" and "non-intuitive independence".

Events are said to be **mutually exclusive** if they have no outcomes in common. In other words, it is impossible that both could occur in a single trial of the experiment. For mutually exclusive events holds $P(A \cdot B) = P(\emptyset) = 0$.

In the die-toss example, events $A = \{3\}$ and $B = \{3,4,5,6\}$ are not mutually exclusive, since the outcome $\{3\}$ belongs to both of them. On the other hand, the events $A = \{3\}$ and $C = \{1,2\}$ are mutually exclusive.

The union $A \cup B$ of two events A and B is an event that occurs if at least one of the events A or B occur. The key word in the definition of the union is **or.**

For mutually exclusive events, the probability that at least one of them occurs is

$$P(A \cup C) = P(A) + P(C)$$

For example, if the probability of event $A = \{3\}$ is $1/6$, and the probability of the event $C = \{1, 2\}$ is $1/3$, then the probability of A or C is

$$P(A \cup C) = P(A) + P(C) = 1/6 + 1/3 = 1/2.$$

The *additivity* property is valid for any number of mutually exclusive events $A_1, A_2, A_3, \ldots$:

$$P(A_1 \cup A_2 \cup A_3 \cup \ldots) = P(A_1) + P(A_2) + P(A_3) + \ldots$$

What is $P(A \cup B)$ if the events A and B are not mutually exclusive.
Example: Two archers problem. (0.7+0.8=1.5).

For any two events A and B, the probability that either A or B will occur is given by the **inclusion-exclusion** rule

$$P(A \cup B) = P(A) + P(B) - P(A \cdot B)$$

Example: Two archers problem. (Correct solution).
The function `archers2`

```
 archers2_function(p, n)
{
        hit <- 0
        prop <- NULL
        for(i in 1:n) {
                Robin <- 0
                Sheriff <- 0
                if(runif(1) < p[1])
                        Robin <- 1
                if(runif(1) < p[2])
                        Sheriff <- 1
                if(Robin + Sheriff > 0)
                        hit <- hit + 1
                prop <- c(prop, hit/i)
        }
        return(prop)
}
```

The function `archers2` gives a sequence ot relative frequencies of hitting the target in n consecutive trials. For $n = 50$ the sequence is:

```
> archers2(c(0.8, 0.7),50)
 [1] 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000
 [8] 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000
```

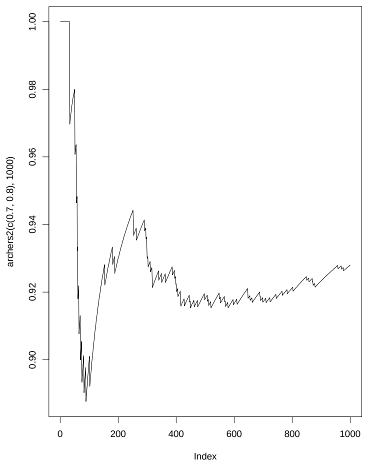

Figure 2: Robin and Sheriff

```
[15] 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000 1.0000000
[22] 0.9545455 0.9565217 0.9583333 0.9600000 0.9615385 0.9629630 0.9642857
[29] 0.9655172 0.9666667 0.9677419 0.9687500 0.9696970 0.9705882 0.9714286
[36] 0.9722222 0.9459459 0.9473684 0.9487179 0.9500000 0.9512195 0.9285714
[43] 0.9302326 0.9318182 0.9111111 0.9130435 0.9148936 0.9166667 0.9183673
[50] 0.9200000
```

After 1000 trials the relative frequency of the event that the target is hit is:

```
> archers2(c(0.8, 0.7), 1000)[1000]
[1] 0.943> archers2(c(0.8, 0.7), 1000)[1000]
[1] 0.943
```

If the events A abd B are exclusive, then $P(A \cdot B) = 0$, and we get the familiar formula $P(A \cup B) = P(A) + P(B)$.

The inclusion-exclusion rule can be generalized to unions of arbitrary number of events. For example, for three events A, Ba and C, the rule is:

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cdot B) - P(A \cdot C) - P(B \cdot C) + P(A \cdot B \cdot C).$$

Example: Three archers problem.

For every event defined on $\mathcal{S}$, we can define a counterpart-event called its **complement**. The complement A^c of an event A consists of all outcomes that are in $\mathcal{S}$, but are *not* in A. The key word in the definition of an complement is **not.** In our example, A^c consists of the outcomes: $\{1, 2, 3, 4, 5\}$.

The events A and A^c are mutually exclusive by definition. Consequently,

$$P(A \cup A^c) = P(A) + P(A^c)$$

Since we also know from the definition of A^c that it includes all the events in the sample space, S, that are not in A, so

$$P(A) + P(A^C) = P(\mathcal{S}) = 1$$

For any complementary events A and A^c, $P(A) + P(A^c) = 1, \quad P(A) = 1 - P(A^c), \quad P(A^c) = 1 - P(A)$

These equations simplify solutions of some probability problems. If $P(A^c)$ is easier to calculate than $P(A)$, then $P(A^c)$ and equations above let us obtain $P(A)$ indirectly.
Example: Three archers problem (Solution by complement).
Example: Reliability of two components in series **Example:** Reliability of two components in paralel

This and some other properties of probability are summarized in table below.

Property	*Notation*
If event S will *always* occur, its probability is 1.	$P(\mathcal{S}) = 1$
If event $\emptyset$ will *never* occur, its probability is 0.	$P(\emptyset) = 0$
Probabilities are always between 0 and 1, inclusive	$0 \leq P(A) \leq 1$
If $A, B, C, \ldots$ are all mutually exclusive then $P(A \cup B \cup C \ldots)$ can be found by addition.	$P(A \cup B \cup C \ldots) = P(A) + P(B) + P(C) + \ldots$
If A and B are mutually exclusive then $P(A \cup B)$ can be found by addition.	$P(A \cup B) = P(A) + P(B)$
Addition rule: The general *addition rule* for probabilities	$P(A \cup B) = P(A) + P(B) - P(A \cdot B)$
Since A and A^c are mutually exclusive and between them include all possible outcomes, $P(A \cup A^c)$ is 1.	$P(A \cup A^c) = P(A) + P(A^c) = P(\mathcal{S}) = 1$, and $P(A^c) = 1 - P(A)$

3 Conditional Probability and Independence

A **conditional probability** is the probability of one event if another event occurred. In the "die-toss" example, the probability of event A, three dots showing, is $P(A) = \frac{1}{6}$ on a single toss. But what if we know that event B, at least three dots showing, occurred? Then there are only four possible outcomes, one of which is A. The probability of $A = \{3\}$ is $\frac{1}{4}$, *given* that $B = \{3, 4, 5, 6\}$ occurred. The *conditional probability of A given B* is written $P(A|B)$.

$$P(A|B) = \frac{P(A \cdot B)}{P(B)}$$

Example: Example with the conditional probs.

Event A is **independent** of B if the conditional probability of A given B is the same as the unconditional probability of A. That is, they are independent if

$$P(A|B) = P(A)$$

In the die-toss example, $P(A) = \frac{1}{6}$ and $P(A|B) = \frac{1}{4}$, so the events A and B are not independent.

The probability that two events A and B will both occur is obtained by applying the **multiplication rule:**

$$P(A \cdot B) = P(A)P(B|A) = P(B)P(A|B)$$

where $P(A|B)$ ($P(B|A)$) means the probability of A given B (B given A).

For independent events only, the equation in the box simplifies to

$$P(A \cdot B) = P(A)P(B)$$

Example: Cards Q and ♠ are independent.
Remark: Independence vs Exclusivness.

The multiplication rule tells us how to find probabilities for composite event $(A \cdot B)$. The probability of $(A \cdot B)$ is used in the general *addition rule* for finding the probability of $(A \cup B)$.

Rule	*Notation*
Definitions	
The *conditional probability* of A given B is the probability of event A, if event B occurred.	$P(A\|B)$
A is *independent* of B if the conditional probability of A given B is the same as the unconditional probability of A.	$P(A\|B) = P(A)$
Multiplication rule:	
The general *multiplication rule* for probabilities	$P(A \cdot B) = P(A)P(B\|A) = P(B)P(A\|B)$
For *independent events* only, the multiplication rule is simplified.	$P(A \cdot B) = P(A)P(B)$

4 Total Probability

Events $H_1, H_2, \ldots, H_n$ form a partition of the sample space $\mathcal{S}$ if
(i) They are mutually exclusive, and
(ii) Their union is the sample space $\mathcal{S}$.

The events $H_1, \ldots, H_n$ are usually called **hypotheses** and from their definition follows that $P(H_1) + \ldots + P(H_n) = 1$ $(= P(\mathcal{S}))$.
Example: Example of a partition of $\mathcal{S}$

Let the event of interest A happens under each of hypotheses H_i with a known (conditional) probability $P(A|H_i)$. Assume, in addition, that the probabilities of hypotheses $H_1, \ldots, H_n$ are known.

Total Probability Formula.

$$P(A) = P(A|H_1)P(H_1) + \ldots + P(A|H_n)P(H_n).$$

Total probability of A is the weighted average of the conditional probabilities $P(A|H_i)$ with weights $P(H_i)$.

Example: Stanley takes a final exam in Statistics by answering 3 questions from an examination card drawn at random from the set of 20 cards. There are 8 favorable cards (Stanley knows answers on all 3 questions). Stanley will get a grade A if he answers all 3 questions. What is the probability for Stanley to get an A if he draws the card

(a) first
(b) second
(c) third?

Solution: Denote with A the event that Stanley draws a favorable card (and consequently gets an A).

(i) If he draws the card first, then $P(A) = 8/20 = 2/5$.

(ii) If Stanley is the second, then a student before him took a card. The card taken might have been favorable (hypothesis H_1) or unfavorable (hypothesis H_2). Obviously, the hypotheses H_1 and H_2 partition the sample space since no other types of cards are possible. Also, the probabilities H_1 and H_2 are $8/20$ and $12/20$. Now, after one card has been taken Stanley draws another. If H_1 had happened, probability of A is $7/19$, and if H_2 had happened, the probability of A is $8/19$. Thus, $P(A|H_1) = 7/19$ and $P(A|H_2) = 8/19$. By the total probability formula, $P(A) = 7/19 \cdot 8/20 + 8/19 \cdot 12/20 = 8/20 = 2/5$.

(iii) Stanley has the same probability of getting an A after two cards have been already taken. The hypotheses are H_1={ both cards taken favorable }, H_2={ exactly one card favorable }, and H_3={ none of the cards taken favorable }. $P(H_1) = 8/20 \cdot 7/19, P(H_3) = 12/20 \cdot 11/19.$ and $P(H_2) = 1 - P(H_1) - P(H_3)$. Next, $P(A|H_1) = 6/18, P(A|H_2) = 7/18$, and $P(A|H_3) = 8/18$. Therefore, $P(A) = 6/18 \cdot 7/19 \cdot 8/20 + 7/18 \cdot \ldots + 8/18 \cdot 11/19 \cdot 12/20 = 12/20$.
Moral: Stanley's lack on the exam does not depend on order of drawing the examination card.

Two-headed coin Out of 100 coins one has heads on both sides. One coin is chosen at random and flipped two times. What is the probability to get

(a) two heads?

(b) two tails?

Solution:

(a) Let A be the event that two heads are obtained. Denote by H_1 the event (hypothesis) that a fair coin was chosen. The hypothesis $H_2 = H_1^c$ is the event taht the two-headed coin was chosen.

$$P(A) = P(A|H_1)P(H_1) + P(A|H_2)P(H_2) = 1/4 \cdot 99/100 + 1 \cdot 1/100 = 103/400 = 0.2575.$$

(b) Exercise. [Ans. 0.2475]

Next we simulate the problem in `S`.

```
> twoheaded
function(n)
{
        proportion <- NULL
        twoheads <- 0
        for(i in 1:n) {
                coin <- "GOOD"
                if(runif(1) < 0.01)
                        coin <- "BAD"
                if(coin == "GOOD" && runif(1) < 0.25)
                        twoheads <- twoheads + 1
                proportion <- c(proportion, twoheads/i)
        }
        return(proportion)
}
> twoheaded(50)
 [1] 0.0000000 0.5000000 0.3333333 0.2500000 0.4000000 0.3333333 0.2857143
 [8] 0.2500000 0.2222222 0.2000000 0.2727273 0.2500000 0.2307692 0.2142857
[15] 0.2000000 0.2500000 0.2941176 0.2777778 0.2631579 0.2500000 0.2380952
[22] 0.2272727 0.2608696 0.2916667 0.2800000 0.2692308 0.2592593 0.2500000
[29] 0.2413793 0.2333333 0.2258065 0.2187500 0.2424242 0.2647059 0.2571429
[36] 0.2500000 0.2432432 0.2368421 0.2307692 0.2500000 0.2439024 0.2619048
[43] 0.2558140 0.2727273 0.2666667 0.2826087 0.2765957 0.2708333 0.2653061
[50] 0.2600000
> twoheaded(2000)[2000]
[1] 0.2585
```

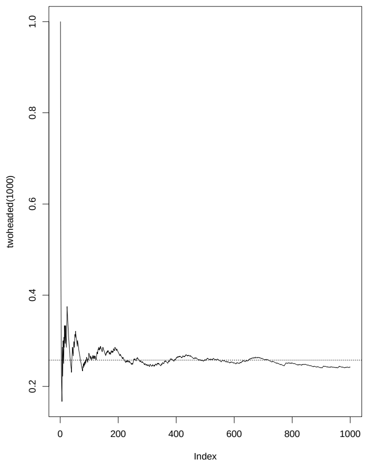

Figure 3: Twoheaded Coin

5 Exercises

Almost one in three Americans is a sports fan, according to a 1990 survey by Lieberman Research for *Sports Illustrated.* About 30% of Americans say they are interested in sports. About 43% are "fairly interested", while 27% are not interested in sports. Based on the study results, what is the probability that: a) One adult interviewed at random will be aery or fairly interested in sports. [Ans: .73]

b) The next three adults interviewed will be sports fans.

c) Two of the next three adults interviewed will not be interested in sports.

d) At most two of the next three adults interviewed will not be interested in sports.

e) Exactly one is very interested in sports iv we know that at most two are very interested in sports. **Sample Space.** An airport limo service has two cars. The larger car can carry

five passengers, the smaller two. Suppose we are interested in the number of passengers each car is carrying. Describe the sample space.

Sample Space. On a surprise quiz with two questions the first question is a multiple choice with a, b, c, and d alternatives while the second question is a *true/false* question. A student is completely unprepared for the quiz (sounds familiar ;-)), and answers at random.

a. Give a sample space for this experiment.

b. The students and the instructor are both interested in the number of points on this quiz. Correct answer on the first question carries 5 points and incorrect -1 point. Correct answer on the second question carries 5 points, incorrect -3. Give a sample space for this experiment and discuss its relation with the sample space from **a.**

Sample Space. A coin is tossed until one head is obtained.

a. Write down a sample space for this experiment.

Let A be the event that at least three tosses are needed to get the first head, B the event that at most 6 tosses are needed.

b. Write down the outcomes for each of the events A and B.

c. Describe (write down the outcomes) the events $A \cdot B$, A^c, B^c, and $A \cup B$.

6. (Prob) In three tosses of a fair coin what is the probability that all faces are the same (all heads or all tails)?

7. (Total) You are on the trip and reach a fork on the road. You know that one route leads you to city A, where the chance of being robbed is 1/3. However, the city A police department is quite efficient, and if you get robbed, there is 9/10 chance that your money will be recovered. The other route leads to the city B, where the chance of being robbed is only 1/5. However in the city B the police department is inept and chance of recovering your money is 1/2.

a. In which city do you have better chance of keeping your money.

b. Suppose you do not know which route leads to which city and so you toss a coin to decide which one to take. What is the probability that you will keep your money?

8. (Prob) A certain bacteria has a 0.4 chance of splitting (into two bacteria) after 1 hour. If the process starts with one bacteria what is the probability that it will be three bacteria after 2 hours?

Odds In eliciting subjective probabilities (in gambling situations, horse races, lotteries, etc) the chance of an event occurring is often expressed in terms of *odds.* If A is an event, the odds in favor of A, written $O(A)$ is defined by $O(A) = P(A) : P(A^c)$. The odds against A are $1 : O(A)$.

For instance, an event A with $P(A) = 1/3$ has odds $O(A) = P(A) : P(A^c) = \frac{1}{3} : \frac{2}{3} = 1 : 2$.

9. If the odds are 3:2 against a favorite horse winning the race, what is the probability of the horse winning the race.

10. **Propagation of genes.** The following example shows how the ideas of independence and conditional probability can be employed in studying the genetic evolution. Consider a single gene which has two forms, *recessive (r)* and *dominant (D)*. Each individual in the population has two genes in his/her chromosomes and thus can be classified into the genotypes DD, RD, and RR. If an individual is drawn at random from the nth generation, then the probabilities of the three genotypes will be denoted by $p_n, 2r_n$, and q_n, respectively. (Clearly, $p_n + q_n + 2\ r_n = 1$)

The problem is to express the probabilities p_n, q_n, and r_n in terms of initial probabilities p_0, q_0, and r_0 and the method of reproduction.

In *Random Mendelian mating* single gene from each parent is selected at random and the selected pair determines the genotype of the offspring. These selections are carried independently of each other from generation to generation. Let M_n be event that R is chosen from the male and F_n be event that R is chosen from the counterpart female. Events M_n and F_n are **independent,** and $P(M_n) = P(F_n) = P(RR)\cdot 1 + P(RD)\cdot 1/2 + P(DD)\cdot 0 = q_n + 2r_n/2 = q_n + r_n$.(by the Total Probability Formula)

Thus,

$$q_{n+1} = P(M_n \cap F_n) = P(M_n) \cdot P(F_n) = (q_n + r_n)^2.$$

Similarly,

$$p_{n+1} = (p_n + r_n)^2,$$

and

$$2r_{n+1} = 1 - p_{n+1} - q_{n+1}.$$

The above equations govern the propagation of genotypes in this population.

Problem. Start with any initial probabilities p_0, q_0 and r_0. (Say, 0.3,0.3, and 0.2; Remember to check: 0.3+0.3+2 0.2=1) Show that $p_1 = p_2, q_1 = q_2$ and $r_1 = r_2$. This stabilizing of probabilities is known as *Hardy-Weinberg law.* It does not hold if other factors (mutation, selection, dependence) are introduced into the model.

11. (Cond) A box contains 50 marbles. Marbles are made of glass (G) or iron (I), and they are colored blue (B) or red (R). The content is described in the table given below:

	G	I	Total
B	10	15	25
R	5	20	25
Total	15	35	50

One marble is selected at random. What is the probability that the marble is

(a) red,

(b) blue **and** made or iron,

(c) red **or** made of glass,

(d) blue, if it is known that it is made of iron.

12. (Cond) A string of Christmas lights contains 4 bulbs. The lights are wired in series, so that if any light fails the whole string go dark. Each light has probability 0.05 of failing during a 3-year period. The lights fail independently of each other. What is the probability that a string of lights will fail in a 3-year period?

13. (Prob) A six sided die has four GREEN and two RED faces, and is balanced so that each face is equally likely to come up. The die will be rolled several times. You mush choose

one of the following three sequences of colors; you will win 25 dollars if the first rolls of the die give the sequence you have chosen.

RGRRR

RGRRRG

GRRRRR

Which sequence should you choose? Explain our choice.

(In a psychological experiment, [3] 63% of 260 students who had not studied probability chose the second sequence. This is the evidence that our intuitive understanding of probability is not always good).

14. (Prob) A baseball player has a batting average of .33. Take this to be the probability that he gets a hit each time he bats.

(a) Find the probability he gets no hits in four times at bat.

(b) Find the probability that the first hit comes in the third bat.

15.(Prob) A five-sided top has sides numbered 1, 2, 3, 4, 5. It is spun and on stopping the side facing downward is noted. Ideally each side has the same chance of being down.

The top is spun twice independently.

(a) What is the probability that the two numbers noted from the spins are different?

(b) What is the probability that the total of the two numbers is 7?

(c) What is the probability that the larger of the two numbers is 3?

16. (Prob) If a card is dealt from a standard pack, what is the probability it will be an honor card (J, Q, K, A) or a spade?

17. (Prob) If a card is drawn from a pack of playing cards, what is the probability it will be an ace or a spade or an honor card?

18. (Total) One box contains four silver and three golden coins, and a second box contains three silver and five golden. One coin is drawn at random from the first box and placed unseen into the second box. What is the probability that the coin now drawn from the second box is golden?

19.(Total) If men constitute 47% of the population and tell the truth 78% of the time, whereas women tell the truth 63% of the time, what is the probability that a person selected at random will answer a question truthfully?

20. (Prob) Player A wins at least one set in the 7 set match against player B, with

[3] Tversky, A. and Kahneman, D. "Extensional vs. intuitive reasoning: the conjunction fallacy in probability judgment." *Psychological Review,* 90, (1983).

probability 0.972. What is the probability that A wins one set?

21. (Prob) In the class of 75 students 12 are smokers, 20 favor Perot, 10 are instate students, 4 instate students smoke, 2 smokers favor Perot, 5 instate students favor Perot, and only one instate student favors Perot and smokes.

One student is chosen at random. Find the probability that

(a) He smokes, but does not favor Perot and is not of state.

(b) He is instate if he smokes.

(c) He favors Perot if he is out of state.

22. (Cond) From the set $\{1, 2, \ldots, 20\}$ one number is drawn at random. Let A be the event: the number is divisible by 2 and B be the event: the number is divisible by 3. Find $P(A|B)$. Discuss independence of events A and B.

Repeat the previous with the set $\{1,2,\ldots, 21\}$.

23. (Total) The class of 10 consists of 4 juniors and 6 seniors. On the oral examination, juniors give satisfactory answer 70% of the time while for seniors this percentage is 75%.

A student is selected at random. What is the probability that he will give a satisfactory answer?

24. (Prob) Find the probability that the electric circuit shown on the graph is working.

25. (Prob) Three archers: Robin Hood, the Sheriff of Nottingham, and King Richard shoot at the target. The probabilities of hitting the target are 0.8, 0.6, and 0.7, respectively. It is assumed that each archer shoots once and independently of others.

Find the following probabilities:

(a) The target is hit.

(b) There are at least two arrows in the target.

(c) There is exactly one arrow in the target.

(d) Only the Sheriff of Nottingham hits the target.

Solution: (a) 0.024, (b) 0.788, (c) 0.188, (d) 0.036. □

26. (Cond) A small village in Transylvania has 134 inhabitants, 56 of which are vampires. Out of 72 village women only 30 are vampires.

You are introduced to a gentleman from the village. What is the probability that the introduced person is a vampire.

Are being a vampire and sex independent?

Solution: $P(V|G) = \frac{26}{62} = 0.42$ (0.4193548387...) Dependent. $P(V|G) = \frac{26}{62} \neq \frac{56}{134} = P(V)$ □

27. (Prob) A gambler has a weighted coin with P(heads) = 2/3. He bets you that in 3 tosses the coin will land heads twice. What is P(he wins)?

28. (Prob) A magician claims to have ESP. If you place 3 cards numbered from 1 to 3 face down, he claims to be able to guess the numbers. Suppose he is really guessing at random. Find the following:

(a) P(he guesses 0 cards correctly)
(b) P(he guesses all 3 cards correctly)
(c) P(he guesses exactly 1 card correctly)
(d) P(he guesses exactly 2 cards correctly)

29. (Prob) In a chicken coop 50% of the chickens are speckled and 50% are plain. A sample of 3 chickens are selected at random with replacement. Find P(the first and third chickens selected are speckled); P(at least two chickens selected are speckled).

30. (Prob) A certain restaurant has good food with probability 2/3 and, independently of that, good service with probability 1/2.

(a) If you go to this restaurant, what is P(you get good food and good service)?
(b) What is P(you get good food or good service)?
(c) If you go to this restaurant twice, what is P(you get good food once and bad service once)?

31. (Cond) In the Land of Utopia it rains on one out of ten days; that is, the probability of rain on any given day is .1. The conditional probability of rain on any one day, given that it rained on the previous day, is .6. Compute the probability that it will rain on two given consecutive days.

32. (Sample Space) An experiment consists of flipping a coin and then flipping it a second time of a head occurs. If a tail occurs on the first flip, then a die is tossed once.

(a) List the elements of the sample space S.

(b) List the elements of S corresponding to event A that a number less than 4 occurred on the die.

(c) Lst the elements of S corresponding to element B that 2 tails occurred.

33. (Cond) A pair of dice is thrown. If it is known that one die shows a 4, what is the probability that:

(a) The other die shows a 5?

(b) The total of both dice is greater than 7?

34. (Total) One bag contains 4 white balls and 3 black balls, and a second bag contains 3 white balls and 5 black balls. One ball is drawn from the first bag and is placed unseen in the second bag. What is the probability that a ball now drawn from the second bag is black?

35. (Cond) Let $P(A) = 0.6, P(B) = 0.4, P(AB) = 0.18$. Find $P(A|B), P(B|A)$.

36. (Prob) A drawer contains 16 socks. 8 of them are brown, 6 are green, and 2 are yellow. Two socks are drawn in succession without replacement. What is the probability the socks will match in color?

37. (Cond) Suppose:

P(rain today)=40%, P(rain tomorrow)=50%, P(rain today and tomorrow)=30%. Given that it rains today, what is the chance that it will rain tomorrow?

38. A probability-inclined executioner offers a convicted murderer a final chance to win his release. He gives the prisoner 8 chips (4 white, 4 black) and 2 indistinguisable urns[4]. He then instructs the prisoner to **place all 8 chips into the 2 urns, with the condition that each urn must contain at least one chip.** He also states that the prisoner can place the chips in any other manner the prisoner so chooses. He than explains that after placement of the chips, he will choose 1 urn ar random, and from the chosen urn, randomly draw one chip. If the chip is white, the prisoner goes free. If chip is black ... the less said the better.

(a) What is the sample space of the prisoners possible allocation scheme?

(Hint: The order of the urns doesn't matter. That is, 4 white in urn 1 and 4 black in urn 2 is the same as 4 white in the urn 2 and 4 black in the urn 1. There are 12 possible allocations.)

(b) Assuming the prisoner wants to live, how should he allocate the chips? What is the maximum probability of survival?

39. In his novel *Bomber*, Len Dieghton argues that a World War II pilot has a 0.02 chance of being shot down on each mission. So in 50 missions, he is "mathematically certain" to be shot down since $50 \times 0.02 = 1$.

[4]There are 3 kinds of urns: Grecian urns, burial urns and probability urns.

a. Assuming the outcome of each mission is independent, is Deighton's reasoning correct?

b. What is the probability of surviving all 50 missions without being shot down?

Sherwood Story. Robin Hood hits the target with the probability 0.8 while the Sheriff of Nottingham hits the target with the probability 0.6. They shoot 3 arrows each and all shots are independent of each other.

(a) What event is more likely:

A: Sheriff hits the target at least 2 times; or **B:** Robin hits the target at most 2 times. Explain.

(b) They shoot in the following order: **S R S R S R.** What is the probability that the Sheriff scores before Robin?

Circuit. Suppose that each of the switches S_i in the following circuit is closed with probability p_i and open with probability $q_i = 1 - p_i$. Calculate the probability that a current will flow through the circuit, assuming that the switches act independently.

p_1	p_2	p_3	p_4	p_5
0.4		0.6	0.8	

q_1	q_2	q_3	q_4	q_5
	0.3			0.5

Simpson, but not O.J. Some special cards are prepared for an unusual game. There are four varieties of cards: cards with red letter A, cards with blue letter A, cards with red letter B and cards with blue letter B. Deck **1** has forty of these cards, and the table below gives the content of this deck.

	Red	Blue
A	5	14
B	5	16

Deck **2** also has forty of these cards, and the table below gives makeup of Deck **2.**

	Red	Blue
A	5	2
B	20	13

Suppose the two decks can be combined to make a single deck of eighty cards.

1. Show:

P(A| Red, Deck **1**) > P(A| Blue, Deck **1**)

P(A| Red, Deck **2**) > P(A| Blue, Deck **2**)

P(A| Red, Combined Deck) < P(A| Blue, Combined Deck)

2. Find:

P(Red | Combined Deck, A)

P(Blue | Deck **1**, B)
3. Are the color and the letter on cards independent in Deck **2**?
Solution: XXXXXXXXXXXXXXXXXXXXXXXXXX

Political Affiliations at Duke University. In their STA110E project Timmy and Scott[5] described the structure of Duke student population with respect to *gender* and *political affili ation.* The following table is their estimate:

	Male	Female
Democrat	11 %	18 %
Independent	16 %	17 %
Republican	23 %	15 %

If you randomly select a Duke student what is the probability that the selected student is:

- (i) Republican and female
- (ii) Republican or female
- (iii) Republican
- (iv) Republican given that it is not Democrat
- (v) Republican given that it is female
- (vi) Female given that it is not Republican.
- (vii) Female given that it is not male and not Republican.
- (viii) Female given that it is not female and Republican.
- (ix) Are the gender and affiliation independent?

[5] Timmy Roach and Scott Wolckenhau er: Political Affiliations at Duke University and in the United States, STA110E Projec t, Fall 1995.