Name: ______________________________

This exam has 5 problems. Each problem is worth 20 points. Points are assigned to parts of a problem as indicated.

This exam is closed book, so please put books and notes on the floor, with the exception of an optional single standard-size formula sheet. You may use a calculator, but you can't share one. A table of the standard normal distribution, a table of some common densities and probability mass functions, and two sheet of scratch paper are included at the end of the exam.

Please show your work, and indicate your answers in the boxes provided. If you need more space for calculations, use the back of the sheet on which the problem appears. If that fills up, then use the scratch pages.

Good Luck!

Please sign the Duke Honor Code:

I have neither given nor received unauthorized aid on this examination.

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signature

Problem	Points
1	(/20)
2	(/20)
3	(/20)
4	(/20)
5	(/20)

1. (20 points) Each of 100 polycarbonate plastic disks in a lot are tested for scratch resistance and shock resistance. The results of these tests are tabulated below:

		Shock Resistance	
		High	Low
Scratch	High	76	12
Resistance	Low	10	2

After testing, one of these disks is chosen at random. Using this information, answer the following questions:

(a) *(5 points) What is the probability that the disk has high shock resistance?*

(5 points) Let A be the event that the disk has high scratch resistance, and let B be the event that the disk has high shock resistance.

$$\mathrm{P}(B) = \frac{76+10}{100} = 0.86$$

(b) *(5 points) What is the probability that the disk has both high shock and high scratch resistance?*

$$\mathrm{P}(A \cap B) = \frac{76}{100} = 0.76$$

(c) *(5 points) You are given that the disk has high scratch resistance. What it the probability that it also has high shock resistance?*

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{0.76}{0.88} = \frac{76}{88} \doteq 0.8636$$

(d) *(5 points) Are the events "Disk has high scratch resistance" and "Disk has high shock resistance" independent? Explain your answer.*

(5 points) No, $\mathrm{P}(A) = (76+12)/100 = 0.88$, $\mathrm{P}(B) = 0.86$. But the product

$$\mathrm{P}(A) \times \mathrm{P}(B) = (0.88)(0.86) = 0.7568$$

does not equal

$$\mathrm{P}(A \cap B) = 0.76,$$

which is a condition which *must* hold for indepdence. Equivalently, you can show that

$$\mathrm{P}(A|B) \neq \mathrm{P}(A),$$

or

$$\mathrm{P}(B|A) \neq \mathrm{P}(B).$$

(The events are "almost independent", however, since 0.7568 is so close to 0.76.)

2. (20 points) In the following situations, an outcome can usefully be modeled by a random variable. For each, indicate which type of random variable is **most appropriate**. (You can assume independence wherever convenient.)

   (a) (5 points) The number of rainy days in the month of March in Durham.
   (a) Poisson (b) binomial (c) geometric (d) uniform (e) exponential (f) normal
   Answer: (b) binomial. Each day has some probability p of rain, and there are $n = 31$ days.

   (b) (5 points) The number of years between 20 inch snowfalls in Durham.

   Answer: (c) geometric. If the probability of a 20 inch snowfal is p, then the probability that we have to wait n years between such a snowfall is $(1-p)^{n-1}p$.

   (c) (5 points) The concentration of cadmium in a sample of sediment from a polluted river.

   Answer: (f) normal. Concentration is a continuous quantity. One can expect the uncertainty in a chemical measurement to be the sum of many small random variables, so the central limit theorem would lead to the selection of the normal.

   (d) (5 points) The number of typos in each 100 pages of an encyclopedia. (a) Poisson (b) binomial (c) geometric (d) uniform (e) exponential (f) normal
   (a) Poisson. There are very few typos in an encyclopedia, and many words. And the probabilities that the typos appear on a particular page are independent. In principle one could also use the binomial, but from what we've discussed in class, the Poisson is clearly *most* appropriate.

3. (20 points) A communications satellite is launched into orbit on an three-stage rocket. The first stage of the rocket fails with probability 0.10, the second and third stages are more reliable, each failing with probability 0.01. The satellite itself has a rocket, which places it into geosynchronous orbit, failing with probability 0.001. Assuming that all stages of the rocket, and the satellite itself, fail (and succeed) independently.

   (a) *(4 points) What is the probability that the satellite is successfully placed into the desired orbit?*
   (4 points) All three stages and the satellite itself must not fail for the setellite to be successfully put into the desired orbit. The event that each of these "components" fails are independent, so the probability of success (the "system reliability") is obtained by multiplying:

$$p = (0.90)(0.99)(0.99)(0.999) \doteq 0.8812$$

   (b) *(4 points) Given that the first stage works successfully, what is this probability?*
   (4 points) Let A be the event that the first stage works, and let B be the event that the satellite is sucessfully put into orbit.

$$\mathrm{P}(B|A) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(A)} = \frac{\mathrm{P}(B)}{\mathrm{P}(A)} \frac{p}{0.90} \doteq 0.9791$$

(c) *(4 points) Given that the first stage does not fail, what is this probability that the second stage does not fail?*

(4 points) The stages succeed or fail independently, so the probability that the second stages works is still .99.

(d) *(8 points) You are told that the satellite did not make it into orbit, but nothing more. What is the probability that the first stage failed?*

(8 points)

$$\mathrm{P}\left(A'|B'\right) = \frac{\mathrm{P}\left(A' \cap B'\right)}{\mathrm{P}\left(B'\right)} = \frac{\mathrm{P}\left(A'\right)}{\mathrm{P}\left(B'\right)} = \frac{0.10}{1-p} \doteq .8418.$$

So you can be fairly certain that the less reliable first stage caused the failure.

4. (20 points) The number of cracks in a section of interstate highway which are large enough to require repair is assumed to be a Poisson random variable with parameter $\lambda = 1$ cracks per mile.

(a) *(5 points) What is the probability that no cracks require repair in 5 miles of highway?*

(5 points) We need the probability that there are no cracks in 5 miles. First, express λ in the appropriate units:

$$\lambda = \frac{1 \text{ crack}}{1mile} = \frac{5 \text{ cracks}}{5miles}$$

Then we calculate the probability that a Poisson random variable with this λ equals 0:

$$p_1 = \mathrm{P}\left(\text{no cracks in 5 miles}\right) = e^{-\lambda} \doteq 0.006738$$

(b) *(5 points) What is the probability that more than one crack requires repair in 5 miles of highway?* If the Poisson probability mass function is $f(x)$, then the answer is

$$p_2 = 1 - f(0) - f(1) = 1 - e^{-5} - 5e^{-5} \doteq 0.9596$$

(c) *(5 points) The highway is inspected and repaired in one mile sections. What is the probability that no cracks are found until mile 5?*

(5 points) The probability that no cracks are found in a one-mile section is

$$p = e^{-1} \doteq 0.3679.$$

The probability that the first crack is in the 5th one-mile section is the probability that a geometric random variable with parameter $1-p$ equals 5:

$$p^4(1-p) \doteq 0.0116$$

(d) *(5 points) What is the probability that 1 of the first 5 sections inspected has cracks, but the other 4 do not?*

(5 points) This is a *binomial* probability with parameters $1-p$ (the probability of at least one crack in a section), and 5 (the number of sections).

$$5p^4(1-p) \doteq 0.0579.$$

5. (20 points) The life of a semiconductor laser is normally distributed with a mean of 7000 hours and a standard deviation of 600 hours.

   Denote the lifetime of a laser by X, and let a standard normal random variable be Z.

   (a) *(5 points) What is the probability that a laser lasts less than 5000 hours?*

   $$\mathrm{P}\left(X \leq 5000\right) = \mathrm{P}\left(7000 + 600Z \leq 5000\right) = \mathrm{P}\left(Z \leq -3.333\right) = 0.000429$$

   (b) *(5 points) What is the probability that a laser lasts more than 9000 hours?*

   (5 points) Since 9000 is as far above the mean as 5000 is below the mean, the answer is the same as in (a) above. Alternatively, you can go through the calculation as above.

   (c) *(5 points) What is the lifetime in hours exceeded by 99% of all lasers?*

   $$\mathrm{P}\left(X \geq t\right) = \mathrm{P}\left(7000 + 600Z \geq t\right) = \mathrm{P}\left(Z \geq \frac{t - 7000}{600}\right) = 0.99$$

   From the table, the 1st percentile of Z is -2.3263. So

   $$\frac{t - 7000}{600} = -2.3263,$$

   and hence $t = 7000 - 600(2.3263) = 5604$ hours.

   (d) *(5 points) Seven lasers are required in an industrial system. What is the probability that all seven function for at least 5000 hours?*

   (5 points) The probability of any one laser functioning this long is $1 - 0.000429 = 0.999571$, using part (a). The probability that all seven function is the probability that a binomial random variable with this value of p and with $n = 7$ is equal to 7:

   $$0.999571^7 = 0.9970$$