Name: ____

This exam has 5 problems. Each problem is worth 20 points. Points are assigned to parts of a problem as indicated.

This exam is closed book, so please put books and notes on the floor, with the exception of an optional single standard-size formula sheet. Also, two pages of some basic formulas is provided at the end of the exam. You may use a calculator, but you can't share one. Tables of the standard normal distribution, Student-t distribution, and two sheet of scratch paper are included at the end of the exam.

Please show your work. If you write an answer in a space other than the space provided for that answer, please label it! If you need more space for calculations, use the back of the sheet on which the problem appears, or else use the scratch pages.

Good Luck!

Please sign the Duke Honor Code:

I have neither given nor received unauthorized aid on this examination.

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signature

Problem	Points
1	(/20)
2	(/20)
3	(/20)
4	(/20)
5	(/20)

1. (a) *(8 points) In a particular hypothesis test, we reject the null hypothesis for large values of a test statistic T. We observe a test statistic $T = 4$ and a P-value of 0.07. In words, what does this P-value mean?*

   (8 points) The smallest value of α for which H_0 can be rejected is 0.07. Alternatively, one can say that the probability of observing a statistic T greater than or equal to 4 if indeed H_0 is true is 0.07.

   (b) *(6 points) A 95% confidence interval for a normal mean μ is $[-2, 1]$. Given this information, consider the following four statements about a test of the hypothesis $H_0 : \mu = 0$ against $H_1 : \mu \neq 0$ at the $\alpha = 0.05$ level. Exactly one of these statements is true; circle the appropriate letter:*

   (a) H_0 is rejected, the P-value is less than 0.05.

   (b) H_0 is rejected, the P-value is greater than 0.05.

   (c) H_0 is not rejected, the P-value is less than 0.05.

   (d) H_0 is not rejected, the P-value is greater than 0.05.

   (6 points) Two observations lead to the selection of the correct answer here. First, $\alpha = 0.05$, so if H_0 is rejected then the P-value must be less than 0.05; conversely if H_0 is not rejected, then the P-value must be greater than 0.05. this gives us "fifty-fifty": the answer is (a) or (d). Second, we notice that a two-sided confidence interval with confidence level $1 - \alpha = 0.95$, includes $\mu = 0$, which is the value under H_0. So a hypothesis test with $\alpha = 0.05$ does not reject H_0, and consequently the answer is (d).

   (c) *(6 points) Assume that thicknesses of playing cards are normally distributed with a mean of 0.25mm and a standard deviation of 0.005mm. What is the distribution, mean, and standard deviation of the thicknesses of decks of 52 cards?*

   (6 points) If X_i is the thickness of a card, $i = 1, \ldots, 52$, and Y is the thickness of a deck, then

$$Y = \sum_{i=1}^{52} X_i.$$

   Since $E(X_i) = \mu$ and $V(X_i) = \sigma^2$, the mean and standard deviation of Y are

$$\mu_Y = 52\mu = 52(0.25) = 13,$$

   and

$$\sigma_Y = \sqrt{52(0.005)^2} = 0.036,$$

   respectively. Finally, since X_i is normal, Y is also normal.

2. *A state is considering raising the maximum speed limit from 55mph to 65mph. A survey is done to see what people think about this. Of $n_1 = 300$ urban residents polled, $x_1 = 63$ are in favor of the proposal. Among rural residents, the corresponding values are $n_2 = 180$ and $x_2 = 50$.*

(a) *(12 points) Test the hypothesis $H_0 : p_1 = p_2$ vs the alternative $H_1 : p_1 \neq p_2$. Let $\alpha = 0.05$. Show the test statistic, region for rejecting H_0, and conclusion of the test.*

(12 points) The test statistic is,

$$Z = \frac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}(1-\hat{p})\frac{1}{n_1} + \frac{1}{n_2}}} = -1.69,$$

where

$$\begin{aligned} \hat{p}_1 &= \frac{x_1}{n_1} = 0.210 \\ \hat{p}_2 &= \frac{x_2}{n_2} = 0.278 \\ \hat{p} &= \frac{x_1 + x_2}{n_1 + n_2} = 0.235. \end{aligned}$$

When H_0 is true, this statistic has (approximately) a standard normal distribution. The rejection region, for $\alpha = 0.05$ is $|Z| > 1.96$, since

$$\mathrm{P}\left(-1.96 \leq Z \leq 1.96\right) \approx 0.95.$$

Since -1.69 is inside this interval, we do not reject H_0. H_0.

(b) *(8 points) What is the P-value for this test?*

$$\mathrm{P}\left(|Z| > 1.69\right) = 0.091.$$

3. *The heart rates of $n = 9$ individuals were measured both after smoking a low dose of marijuana and after smoking tobacco. Assume that the heart rate difference for each individual is normally distributed with mean μ. The sample mean and standard deviation of the 9 observed differences are $\bar{x} = 8.2$ and $s = 12.8$.*

   (a) *(12 points) Test $H_0 : \mu = 0$ against $H_1 : \mu > 0$. Use $\alpha = 0.05$. Show the test statistic, critical region, and test conclusion.*

   (12 points) The test statistic

$$T = \frac{\bar{x}}{s/\sqrt{n}} = 1.922$$

   has a t-distribution with $n - 1 = 8$ degrees of freedom. For a one sided test we reject for large values of T. From the table, we see that the appropriate rejection region for $\alpha = 0.05$ is 1.860. Since 1.922 is greater than 1.860, we reject H_0.

   (b) *(8 points) Construct a 95% lower confidence interval for μ.*

   (8 points) A 95% lower confidence limit for mu is

$$L = \bar{x} - ts/\sqrt{n} = 8.2 - (1.860)12.8/3 = 0.266.$$

4. *Soil Ph was measured for $n = 9$ sample at each of two locations. The sample means and standard deviations are $\bar{x}_1 = 8.03$, $\bar{x}_2 = 7.44$, $s_1 = 0.29$, $s_2 = 0.22$. Assume that Ph is normally distributed, with means μ_1 and μ_2 for the two locations.*

   (a) *(12 points) Test the hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative $H_1 : \mu_1 \neq \mu_2$, with $\alpha = 0.05$. Show the test statistic, region for rejection of H_0, and conclusion of your test. (To keep things simple, you may assume that the unknown variances are equal.)*

   (12 points) The pooled standard deviation is

$$s_p = \sqrt{\frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}} = 0.257.$$

   The test statistic is

$$T = \frac{\bar{x}_1 - \bar{x}_2}{s_p\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} = 4.86.$$

   We reject H_0 when a t-distributed random variable with $n_1 + n_2 - 2 = 16$ degrees of freedom exceeds 2.120 in absolute value. Since 4.86 is greater than 2.12 we can easily reject H_0.

   (b) *(8 points) Construct a 95% two-sided confidence interval for $\mu_1 - \mu_2$.* The confidence interval is

$$\bar{x}_1 - \bar{x}_2 \pm t_{n_1+n_2-2,\alpha/2} s_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}} = 0.590 \pm 0.2584$$

5. *The joint probability mass function of two random variables, X and Y is specified by the following table:*

x	y	$f_{X,Y}(x, y)$
1	1	0.2
1	2	0.2
1	3	0.2
2	1	0.2
2	2	0.1
3	2	0.1

   (a) *(8 points) What is $E(X)$?*

$$E(X) = (1)(0.6) + (2)(0.3) + (3)(0.1) = 1.5$$

   (b) *(8 points) What is $E(Y)$?*

$$E(Y) = (1)(0.4) + (2)(0.4) + (3)(0.2) = 1.8$$

   (c) (4 points) To show independence, we need to show that for *all* values x and y,

$$\mathrm{P}(X = x \cap Y = y) = \mathrm{P}(X = x)\mathrm{P}(Y = y).$$

To show dependence, we need only show that this relation shows for *some* x and y.

$$\begin{aligned} \mathrm{P}\left(X=1 \cap Y=1\right) &= 0.2 \\ \mathrm{P}\left(X=1\right)\mathrm{P}\left(Y=1\right) &= (0.6)(0.4)=0.24. \end{aligned}$$

Since these values are not equal, X and Y are not independent.