1. A chemist is interested in the concentration of glucose in human serum. She makes 10 independent glucose measurements, on samples of serum which can each be assumed to have the same mean concentration μ. The average of these measurements (in micrograms per ml) is $\bar{x} = 78$. From extensive previous work, the chemist knows that the standard deviation of the distribution of a measurement is $\sigma = 3$.

   (a) (6 points) What is the approximate sampling distribution of the random variable $\bar{X}$, which corresponds to the average of the measurements? Provide the distribution type (normal, binomial, etc.), mean, and standard deviation.

   •The sampling distribution of the sample mean is approximately normal, with mean equal to the population mean (μ), and variance equal to the population variance ($\sigma^2/n = 9/10$). Many (most?) said that the mean of the sampling distribution was equal to the sample average, $\bar{x}$. This is wrong: $\bar{x} = 78$ is a sample value from the sampling distribution, the one we happened to get from the data.

   (b) (4 points) We do not know the precise population from which these measurements are a random sample. Why do you think the approximation that you used above is likely to be a good one?

   •The central limit theorem tells us that $\bar{X}$ will be approximately normally distributed, even for quite small sample sizes, for "most" populations.

   (c) (8 points) What is the P-value of a test of the null hypothesis $H_0 : \mu = 75$ against the alternative $H_1 : \mu > 75$.

   •The P-value is the probability of observing a test statistic as large or larger than the one that occurred, if one assumes that the null hypothesis is true:

$$p = \mathrm{P}\left(\frac{\bar{X} - 75}{\sigma/\sqrt{n}}\right) \geq \frac{78 - 75}{3/\sqrt{10}} = 1 - \Phi(3.2) = .0007$$

   (d) (2 points) For the above hypothesis test, what is the smallest Type I error (α) for which H_0 can be rejected?

   •By definition, the P-value is the smalles α for which H_0 can be rejected. So the answer is $\alpha = p = .0007$.

2. A total of 30 measurements are made on cyanide in soil. The sample mean and standard deviation of these measurements are $\bar{x} = 95$mg/kg and $s = 20$mg/kg.

   (a) (10 points) Construct a 90% two-sided confidence interval for μ, the population mean concentration of cyanide.

   •The test statistic

$$\frac{\bar{X} - \mu}{S/\sqrt{n}}$$

   has a t-distribution with $30 - 1 = 29$ degrees of freedom. For a 90% confidence interval, we need $t_{29,.10/2} = 1.699$ (i.e., two-sided interval, .05 in each tail). The confidence interval is

$$\bar{x} \pm 1.699 s/\sqrt{n} = [88.8\,101.2]$$

(b) Perform the hypothesis test for which $H_0 : \mu = 100$, and $H_1 : \mu < 100$. Use $\alpha = 0.05$.

i. (3 points) What is the observed standardized test statistic under H_0?

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$$t_{29} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{95 - 100}{20/\sqrt{30}} = -1.37$$

ii. (5 points) For what values of this statistic should you reject the null hypothesis?

•We're doing a one-sided test, and we reject the null hypothesis for small values of the test statistic. So we use the t value which "cuts off" .05 in the lower tail, i.e., $t_{29,.05} = -1.699$. We reject the null hypothesis for standardiized test statistics ≤ 1.699.

iii. (2 points) Can you reject the null hypothesis for this problem?

•No, since $-1.37 > -1.699$.

3. The breaking strength of a yarn used in manufacturing drapery material is required to be at least 100psi. A random sample of 9 specimens in tested, and the average breaking strength is found to be $\bar{x} = 98$psi. From past experience, we know that the standard deviation of breaking strength for this yarn is $\sigma = 2$psi. We are interested in hypothesis tests and confidence intervals for the population mean breaking strength, μ.

(a) Test the hypothesis $H_1 : \mu > 100$, where the null hypothesis is $H_0 : \mu = 100$. Use $\alpha = 0.05$.

i. (4 points) What is the observed standardized test statistic under the null hypothesis?

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$$z_0 = \frac{98 - 100}{2/\sqrt{9}} = -3$$

ii. (4 points) What is the critical region for rejecting the null hypothesis?

•One sided test, where we reject H_0 for large values of the standardized test statistic. Since $\alpha = 0.05$ and the test statistic has a standardized normal distribution, we reject for values of the standardized test statistic greater than $z_{.95} = -1.65$.

iii. (2 points) Can you reject the null hypothesis for this problem?

•No, since $-1.65 > -3$.

(b) (10 points) Find a 95% two-sided confidence interval for μ.

•For a two sided interval with confidence 95%, there will be 2.5% in each tail. So we need to use the normal quantile $z_{1-.05/2} = z_{.975} = 1.96$. The confidence interval is

$$\bar{x} \pm z_{.975}\sigma/\sqrt{n} = 98 \pm (1.96)(2/\sqrt{9}) = [96.7\,, 99.3]$$

4. Scientists believe that frogs eggs are less likely to hatch in sunlight than in shade, because of damage done by ultraviolet radiation. California tree frogs, on the other hand, have an enzyme which is believed to protect the eggs from the damaging effects of sunlight.

An experiment was conducted with California tree frogs. Of 70 eggs in the shade, 34 hatched. Of 80 eggs in the sun, 31 hatched.

•Let p_1 be the probability that an egg hatches in sunlight, and let p_2 be the probability that it hatches in shade. Assume that the number of eggs hatching are binomial random variables, and use the normal approximation to the binomial distribution.

(a) i. (5 points) Estimate the difference in probabilities $p_2 - p_1$ ("shade − sun")

•We estimate each probability by the observed proportion of hatched eggs: $\hat{p}_1 = 31/80$, $\hat{p}_2 = 34/70$, so the estiamte of the difference is

$$\hat{p}_2 - \hat{p}_1 = \frac{34}{70} - \frac{31}{80} = 0.098$$

ii. (5 points) What is the standard error of this estimate?

•The estimated variance of $\hat{p}_1$ is

$$\frac{\hat{p}_1(1-\hat{p}_1)}{n_1} = \frac{0.39(1-0.39)}{80} = 0.00297$$

The estimated variance of $\hat{p}_2$ is

$$\frac{\hat{p}_2(1-\hat{p}_2)}{n_2} = \frac{0.49(1-0.49)}{70} = 0.00357$$

The standard error is the standard deviation of the estimate. The variance of $\hat{p}_2 - \hat{p}_1$ is the sum of the variances. To get the standard error, we take the square root:

$$\mathrm{SE} = \sqrt{0.00297 + 0.00357} = 0.08$$

(b) (8 points) Construct a 95% (one-sided) lower confidence interval for $p_2 - p_1$.

•Our estiamte of $p_2 - p_1$ has an approximate normal distribution, so we use $z_{0.05} = -1.65$. With 95% confidence, the population value of $p_2 - p_1$ is greater than

$$\hat{p}_2 - \hat{p}_1 - 1.65\mathrm{SE} = 0.098 - 1.65(0.08) = -0.03$$

(c) (2 points) Based on this confidence interval, do you think that one should conclude that eggs are more likely to hatch in shade than in sun?

•No, because the confidence interval includes 0, so at the 95% confidence level we can't say that $p_2 - p_1$ is positive.

5. Wear tests were performed on rubber for an automotive application which is manufactured by two different companies. The decrease in mass due to wear of 25 samples from each manufacturer was measured.

•For manufacturer 1, the sample mean and standard deviation of mass loss due to wear are $\bar{x}_1 = 20$ and $s_1 = 2$, respectively. For manufacturer 2, the corresponding results are $\bar{x}_2 = 15$ and $s_2 = 8$. When answering the questions below, assume that the population variances for the mass losses for the two manufacturers are unknown and unequal.

(a) i. (2 points) Estimate the difference $\mu_1 - \mu_2$ of the mass loss for the two manufacturers.

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$$\hat{\mu}_1 - \hat{\mu}_2 = 20 - 15 = 5$$

ii. (3 points) What is the standard error of this estimate?

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$$s = \sqrt{s_1^2/n_1 + s_2^2/n_2} = \sqrt{4/25 + 64/25} = 1.65$$

(b) Test the hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative $H_1 : \mu_1 \neq \mu_2$. Use $\alpha = 0.05$.

i. (3 points) What is the observed standardized test statistic under H_0?

•Since we are told that we cannot assume that the variances are equal, we should not use the pooled variance estimate s_p, but rather the standard error of the mean from the previous part of the problem. Under H_0 $\mu_1 = \mu_2$, and the test statistic is

$$t_\nu = \frac{\bar{x}_1 - \bar{x}_2}{s} = \frac{5}{1.645} = 3.03$$

ii. (10 points) For what values of this statistic should you reject H_0?

•The test statistic above has a t distribution, since the variances are estimated. Also, since the population variances are not assumed equal, we should estimate the degrees of freedom as

$$\nu = \frac{\left(s_1^2/n_1 + s_2^2/n_2\right)^2}{\frac{(s_1^2/n_1)^2}{n_1+1} + \frac{(s_2^2/n_2)^2}{n_2+1}} - 2 = \frac{1.645}{\frac{(4/25)^2}{26} + \frac{(64/25)^2}{26}} - 2 \approx 27$$

Since wer're doint a two-sided test, we reject if the absolute value of the test statistic is greater than $t_{27,.025} = 2.052$.

iii. (2 points) Can you reject H_0 for this problem?

•Yes, $3.03 > 2.052$.