STA 205 - Probability and Measure Theory

Homework 1

1. Consider a sample space Ω.
(a) Prove that any intersection of σ-fields (of subsets of Ω) is a σ-field.
(b) Show by example that a union of σ-fields may not be a σ-field.

2. Consider a sample space Ω and collections $\mathcal{E}_1$ and $\mathcal{E}_2$ of subsets of Ω. Prove the following:
(a) If $\mathcal{E}_1 \subseteq \mathcal{E}_2$ then $\sigma(\mathcal{E}_1) \subseteq \sigma(\mathcal{E}_2)$.
(b) If $\mathcal{E}_1$ is a σ-field then $\sigma(\mathcal{E}_1) = \mathcal{E}_1$.
(c) If $\mathcal{E}_1 \subseteq \sigma(\mathcal{E}_2)$ and $\mathcal{E}_2 \subseteq \sigma(\mathcal{E}_1)$ then $\sigma(\mathcal{E}_1) = \sigma(\mathcal{E}_2)$.

3. (Billingsley, problem 2.3) Let $\mathcal{F}$ be a collection of subsets of a sample space Ω.
(a) Suppose that $\Omega \in \mathcal{F}$ and that $A, B \in \mathcal{F}$ implies $A \cap B^c \in \mathcal{F}$. Show that $\mathcal{F}$ is a field.
(b) Suppose that $\Omega \in \mathcal{F}$ and that $\mathcal{F}$ is closed under the formation of complements and finite disjoint unions. Show that $\mathcal{F}$ need not be a field.

4. A Borel subset of $\overline{R}^+$ was characterized as being equal to either a Borel subset of R^+ or to the union of a Borel subset of R^+ and $\{\infty\}$. We used the lemma that describes σ-fields for subspaces A of a sample space Ω through the σ-field of the entire space Ω, taking $\Omega = \overline{R}^+$ and $A = R^+$. Obtain this result employing the same lemma in a different fashion.

5. Consider a sequence $\{A_n\}_{n=1}^{\infty}$ of subsets of a sample space Ω. The sequence is said to be increasing or decreasing, respectively, if the corresponding sequence of indicator functions is increasing or decreasing. Prove that $lim_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} A_n$, if $\{A_n\}_{n=1}^{\infty}$ is increasing and $lim_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} A_n$, if $\{A_n\}_{n=1}^{\infty}$ is decreasing.

6. Consider sequences $\{A_n\}_{n=1}^{\infty}$, $\{B_n\}_{n=1}^{\infty}$ and $\{C_n\}_{n=1}^{\infty}$ of subsets of a sample space Ω. Suppose that there exists a postitive integer k such that $A_n \subseteq B_n \subseteq C_n$ for all $n \geq k$. Assume also that $limsup_{n\to\infty} C_n \subseteq liminf_{n\to\infty} A_n$. Prove that each of $lim_{n\to\infty} A_n$, $lim_{n\to\infty} B_n$ and $lim_{n\to\infty} C_n$ exists and that all three limits are the same.

7. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Show that

$$\mathcal{P}(limsup_{n\to\infty} A_n) \geq limsup_{n\to\infty} \mathcal{P}(A_n) \geq liminf_{n\to\infty} \mathcal{P}(A_n) \geq \mathcal{P}(liminf_{n\to\infty} A_n).$$

8. We define two events A and B in a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ to be equal almost surely (a.s.) if $\mathcal{P}(A \cap B^c) = 0$ and $\mathcal{P}(B \cap A^c) = 0$. Prove the following:
(a) If A and B are equal a.s., $\mathcal{P}(A) = \mathcal{P}(B)$.
(b) A and B are equal a.s. if and only if $\mathcal{P}(A \cap B) = max\,\{\mathcal{P}(A), \mathcal{P}(B)\}$.

9. Consider a measurable space $(\Omega, \mathcal{F})$ and a set function $\mathcal{P}$: $\mathcal{F} \longrightarrow [0,1]$ which satisfies $\mathcal{P}(\Omega) = 1$ and $\mathcal{P}(A \cup B) = \mathcal{P}(A) + \mathcal{P}(B)$ for any A and B in $\mathcal{F}$ with $A \cap B = \emptyset$. Prove that $\mathcal{P}$ is countably additive (i.e., $\mathcal{P}$ is a probability measure on $(\Omega, \mathcal{F})$) if and only if $\mathcal{P}$ is continuous (i.e., $\mathcal{P}(lim_{n\to\infty} A_n) = lim_{n\to\infty} \mathcal{P}(A_n)$, for any sequence $\{A_n\}_{n=1}^{\infty}$ of sets in $\mathcal{F}$ for which $lim_{n\to\infty} A_n$ exists.)