STA 205 - Probability and Measure Theory

Homework 2

1. Assuming the usual Borel σ-field on R, prove that any nondecreasing function from R to R is measurable.

2. Let X and Y be R-valued random variables defined on the same probability space $(\Omega, \mathcal{F}, \mathcal{P})$. We have shown that $A = \{\omega \in \Omega : X(\omega) \neq Y(\omega)\}$ is an event in $\mathcal{F}$. Assume that $P(A) = 0$. Prove that the distributions of X and Y are equal, i.e., $P(X^{-1}(B)) = P(Y^{-1}(B))$ for any Borel subset B of R.

3. If $\{X_n\}_{n=1}^{\infty}$ is any sequence of R-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$, then $lim_{n\to\infty} X_n = 0$, with probability 1, if and only if $P(limsup_{n\to\infty} A_{n,k}) = 0$, for all $k =$ 1,2,..., where $A_{n,k} = \{|X_n| > k^{-1}\}$.

4. A sequence $\{X_n\}_{n=1}^{\infty}$ of R-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ is said to converge completely if for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(|X_n| > \varepsilon) < \infty$. Show that if $\{X_n\}_{n=1}^{\infty}$ converges completely then $lim_{n\to\infty} X_n = 0$, with probability 1.

5. Consider a sequence $\{X_n\}_{n=1}^{\infty}$ of random variables defined on a common probability space. Assume that each random variable X_n is uniformly distributed on [0,1]. Prove that for all $a > 1$, $lim_{n\to\infty}(n^a X_n)^{-1} = 0$, with probability 1.

6. [Delta distribution or unit point mass] If X, defined on some probability space, is equal to a constant $a \in R$, obtain the distribution function F of X.

7. Let X be an R-valued random variable defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$. If F is the distribution function of X prove that for all real numbers $a < b$, $P(\{\omega \in \Omega : X(\omega) \in (a, b]\}) = F(b) - F(a)$. Find analogous formulas involving intervals of the form (a, b), $[a, b)$ and $[a, b]$. For $x \in R$, show that $P(\{\omega \in \Omega : X(\omega) = x\}) = F(x) - F(x-)$. As a consequence, conclude that if F is continuous all the events $\{\omega \in \Omega : X(\omega) = x\}$ have probability 0.

8. Prove that if X is a simple random variable defined on some probability space $(\Omega, \mathcal{F}, \mathcal{P})$ with distribution function F, then $E(X)$ can be written in the form $E(X) = \sum_{x\in R} x(F(x) - F(x-))$, the summation being meaningful since there are only finitely many nonzero terms.

9. Let X and Y be random variables defined on a common probability space $(\Omega, \mathcal{F}, \mathcal{P})$ and taking values on the extended positive real line. For any nonnegative real constants a and b show that $E(aX + bY) = aE(X) + bE(Y)$, whether finite or infinite, provided that $0 \times \infty$ and $\infty \times 0$ are understood to equal 0.