STA 205 - Probability and Measure Theory

Homework 4

1. Let $F(x) = x/3$, if $0 \leq x < 1$, $F(x) = 1/2$, if $1 \leq x < 2$ and $F(x) = x/3$, if $2 \leq x \leq 3$. Prove that $\int_0^3 F(x)dF(x)$ does not exist.

2. Consider reals a_i, p_i, $i = 1, ..., n$, such that $0 < a_1 < ... < a_n < 1$ and $p_i > 0$, $i = 1, ..., n$, with $\sum_{i=1}^n p_i = 1$. Define
$$F(x) = \sum\nolimits_{i=1}^n p_i 1_{[a_i,\infty)}(x).$$
(a) Show that F is a distribution function.
(b) For a continuous $\phi : [0, 1] \to R$, find $\int_0^1 \phi(x)dF(x)$ (express it in terms of the p_i's and the $\phi(a_i)$'s.)

3. Consider an increasing sequence $\{X_n\}_{n=1}^{\infty}$ of $\overline{R}^+$-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. Set $X = lim_{n\to\infty} X_n$. If $E(X) < \infty$, then prove that $Var(X) = lim_{n\to\infty} Var(X_n)$.

4. For $z > 0$, use the Markov inequality (the general version given in class) to obtain an upper bound for $P(|X| \geq z)$ in terms of $E(|X|)$ and another in terms of $E(X^2)$, and also an upper bound for $P(X \geq z)$ in terms of $E(X^+)$.

5. Let $\{A_j\}_{j=1}^{\infty}$ be a sequence of events in a probability space $(\Omega, \mathcal{F}, P)$ and let $\{a_j\}_{j=1}^{\infty}$ be a sequence of real numbers. Assume that
$$\sum\nolimits_{j=1}^{\infty} |a_j| \, P(A_j) < \infty.$$
Prove that
$$X(\omega) = \sum\nolimits_{j=1}^{\infty} a_j 1_{A_j}(\omega)$$
is an almost surely defined random variable and that
$$E(X) = \sum\nolimits_{j=1}^{\infty} a_j P(A_j).$$