STA 205 - Probability and Measure Theory

Homework 5

1. Let X be an R-valued random variable, defined on a probability space $(\Omega, \mathcal{F}, P)$, having finite mean μ and finite standard deviation σ. Prove that for $0 \leq z \leq \sigma$,

$$P(\{\omega \in \Omega : |X(\omega) - \mu| \geq z\}) \geq \frac{\sigma^4(1-(z/\sigma)^2)^2}{E(|X-\mu|^4)}.$$

2. Let X be an R-valued random variable, defined on some probability space $(\Omega, \mathcal{F}, P)$ and let $\varphi : R \to R$ be a nondecreasing function. Assume that X and $\varphi \circ X$ both have finite variance. Show that $Cov(X, \varphi \circ X) \geq 0$.

3. Consider a sequence $\{A_n\}_{n=1}^{\infty}$ of events in a probability space $(\Omega, \mathcal{F}, P)$. Assume that all of the events A_n have the same probability p. For n = 1,2,..., let $M_n(\omega)$ be the number of events in the finite sequence $\{A_j\}_{j=1}^{n}$ which contain the sample point ω. Find conditions on the quantities $P(A_i \cap A_j)$, for $i \neq j$, that imply $lim_{n\to\infty} P(\left\{\omega \in \Omega : \left|\frac{M_n(\omega)}{n} - p\right| \geq \varepsilon\right\}) = 0$, for all $\varepsilon > 0$.

4. Show that if φ is an R-valued function which is convex or concave on an interval $J \subseteq R$, then φ is measurable.

5. Let X be an R-valued random variable defined on a probability space $(\Omega, \mathcal{F}, P)$. Prove that the function $f(p) = (E(|\ X|^p))^{1/p}$, with $0 < p < \infty$, is a nondecreasing $\overline{R}$-valued function.

6. Given a space Ω and two Sierpiński classes $\mathcal{E}_1$ and $\mathcal{E}_2$ of subsets of Ω, prove that $\mathcal{E}_1 \cap \mathcal{E}_2$ is also a Sierpiński class. Show by a counter-example that $\mathcal{E}_1 \cup \mathcal{E}_2$ may not be a Sierpiński class.

7. Consider a field $\mathcal{E}$ of subsets of a set Ω and let T be a nonnegative countably additive function defined on $\mathcal{E}$ such that $T(\Omega) = 1$. Let $\{A_n\}_{n=1}^{\infty}$ be a decreasing sequence of sets in $\mathcal{E}$ such that $lim_{n\to\infty} A_n = \emptyset$. Prove that $lim_{n\to\infty} T(A_n) = 0$.

8. Let $(\Omega, \mathcal{F}, \mu)$ be a measure space, and let f be an $\overline{R}^+$-valued measurable function defined on Ω. Let $B = \{\omega \in \Omega : f(\omega) > 0\}$, and define $\nu(A) = \mu(A \cap B)$ for all $A \in \mathcal{F}$. Prove that ν is a measure on $(\Omega, \mathcal{F})$ and that $\int f d\mu = \int f d\nu$. Moreover, if $\int f d\mu < \infty$, show that $(\Omega, \mathcal{F}, \nu)$ is a σ-finite measure space.