STA 205 - Probability and Measure Theory

Homework 6

1. Let $\{A_n\}_{n=1}^{\infty}$ be an increasing sequence of measurable sets in a measure space $(\Omega, \mathcal{F}, \mu)$. Prove that $lim_{n\to\infty}\mu(A_n) = \mu(lim_{n\to\infty}A_n)$.

2. Let $\{A_n\}_{n=1}^{\infty}$ be any sequence of measurable sets in a measure space $(\Omega, \mathcal{F}, \mu)$. Show that $\mu(liminf_{n\to\infty}A_n) \leq liminf_{n\to\infty}\mu(A_n)$. In addition, show by example that $limsup_{n\to\infty}\mu(A_n) \leq \mu(limsup_{n\to\infty}A_n)$ does not hold in the general measure space setting. (Hint: work with Lebesgue measure on R.)

3. Let $\{A_n\}_{n=1}^{\infty}$ be a sequence of measurable sets in a measure space $(\Omega, \mathcal{F}, \mu)$ such that $lim_{n\to\infty}A_n$ exists and $\mu(\bigcup_{n=1}^{\infty} A_n) < \infty$. Prove that $lim_{n\to\infty}\mu(A_n) = \mu(lim_{n\to\infty}A_n)$.

4. Consider Lebesgue measure λ on R with the usual Borel σ-field. Provide a sequence $\{A_n\}_{n=1}^{\infty}$ of Borel subsets of R that satisfies $\lambda(A_i \cap A_j) \leq \lambda(A_i)\lambda(A_j)$, for all i, j with $i \neq j$, and $\sum_{n=1}^{\infty} \lambda(A_n) = \infty$, but $\lambda(limsup_{n\to\infty}A_n) \neq 1$, proving therefore that the Borel-Cantelli lemma does not hold in general for infinite measure spaces.

5. Consider the Lebesgue integrals $L_n = \int_0^{\infty} exp(-x)sin(nx/(n^2+1))dx$, $n =$ 1,2,.... Using an appropriate convergence theorem, show that $lim_{n\to\infty}L_n = 0$. (Here, the underlying measure is Lebesgue measure and you can assume, as the expression for L_n indicates, that all Lebesgue integrals can be evaluated as Riemann integrals.)

6. Consider a sequence $\{X_n\}_{n=1}^{\infty}$ of R^+-valued random variables defined on a common probability space $(\Omega, \mathcal{F}, P)$. We know (Homework 2, problem 4) that if for any $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(X_n > \varepsilon) < \infty$ then $lim_{n\to\infty}X_n = 0$, with probability 1. Give an example of a sequence $\{X_n\}_{n=1}^{\infty}$ of R^+-valued random variables ($X_n \geq 0$, for all n) that satisfies for all $\varepsilon > 0$, $\sum_{n=1}^{\infty} P(X_n > \varepsilon) < \infty$, but for which $lim_{n\to\infty}E(X_n) \neq 0$.

7. Let $\{A_n\}_{n=1}^{\infty}$ be an independent sequence of events in a probability space $(\Omega, \mathcal{F}, P)$. Prove that $P(\bigcap_{n=1}^{\infty} A_n) = \prod_{n=1}^{\infty} P(A_n)$.

8. Consider random variables $X_k : (\Omega, \mathcal{F}, P) \to (\Psi_k, \mathcal{G}_k)$ and measurable functions $\varphi_k : (\Psi_k, \mathcal{G}_k) \to (\Theta_k, \mathcal{H}_k)$, $k =$ 1,2,.... Prove that if the sequence $\{X_k\}_{k=1}^{\infty}$ is independent then the sequence $\{\varphi_k \circ X_k\}_{k=1}^{\infty}$ is independent.

9. Consider $\{A_n\}_{n=1}^{\infty}$ and $\{B_n\}_{n=1}^{\infty}$ two sequences of events in a probability space $(\Omega, \mathcal{F}, P)$ such that for each n, A_n and B_n are independent. Assuming that $A = lim_{n\to\infty}A_n$ and $B = lim_{n\to\infty}B_n$ exist, show that A and B are independent.