STA-226, Spring Problem set 2. Due Thursday Feb 1st.

1. Consider the following decision table (loss function), in which the consequences are monetary payoffs.

		States of nature		
		θ_1	θ_2	θ_3
	a_1	£100	£110	£120
Actions	a_2	£90	£100	£120
	a_3	£100	£110	£100

convert this problem into one of choosing between lotteries, as in the von Neumann – Morgenstern theory. The decision maker holds the following indifferences:

£100 $\sim$ £120 w.p 1/2; £90 w.p 1/2;

£110 $\sim$ £120 w.p 4/5; £90 w.p 1/5;

Assume that $P(\theta_1) = P(\theta_3) = 1/4$ and $P(\theta_2) = 1/2$.
Which action should the decision maker choose? *From French Chapter 5 — problem 3.*

2. A forecaster must announce probabilities $\boldsymbol{a} = (a_1, ...a_k)$ for the events $E_1, ...E_k$. These events form a partition, that is one and only one of them will occur. The forecaster will be scored based on the loss function

$$L(E_j, \boldsymbol{a}) = \sum_{i=1}^{k} |\boldsymbol{a}_i - I_{i=j}|.$$

Here $I_{i=j}$ is 1 if $i = j$ and 0 otherwise. Let $\pi = (\pi_1, \ldots, \pi_k)$ represent the forecasters own belief about $E_1, ...E_k$. Show that this scoring rule in not proper, that is show that there exist an $a \neq \pi$ such that:

$$\sum_{j=1}^{J} L(E_j, \boldsymbol{a})\pi_j < \sum_{j=1}^{J} L(E_j, \pi)\pi_j.$$

Because you are looking for a counterexample, it is okay to consider a simplified version of the problem, for example by picking a small J.

3. (DeGroot, 1970). This is a very nice example, and we will return to it when we will study the Ellsberg paradox. The idea there is that in the situation of case (a), according to the Bayes rule, if $E\{\theta\} = 1/2$ you should be indifferent between the two boxes. But empirically, most people still prefer box 1. What do you think?

Consider two boxes, each of which contains both red balls and green balls. It is known that one-half of the balls in box 1 are red and the other half are green. In box 2, the proportion θ of red balls is not known with certainty, but this proportion has a probability distribution over the interval $0 \leq \theta \leq 1$.

(a) Suppose that a person is to select a ball at random from either box 1 or box 2. If that ball is red, he wins \$1; it it is green he wins nothing. Show that under any utility function which is an increasing function of monetary gain, the person should prefer selecting the ball from box 1 if, and only if, $E\{\theta\} < 1/2$.

(b) Suppose that a person can select n balls $(n \geq 2)$at random from either of the boxes, but that all n balls must be selected from the same box; suppose that each selected ball will be put back in the box before the next ball is selected; and suppose that he will receive \$1 for each red ball selected and nothing for each green ball. Also, suppose that his utility function u of monetary gain is strictly concave over the interval $[0, n]$, and suppose that $E\{\theta\} = 1/2$. Show that person should prefer to select the balls from box 1.

Hint: Show that if the balls are selected from box 2, then for any given value of θ, $E\{u|\theta\}$ is a concave function of θ on the interval $0 \leq \theta \leq 1$. This can be done by showing that:

$$\frac{d^2}{dx^2}E\{u|\theta\} = n(n-1)\sum_{i=0}^{n-2}[u(i) - 2u(i+1) + u(i+2)]\binom{n-2}{i}x^i(1-x)^{n-2-i} < 0.$$

Then apply Jensen's inequality to $g(\theta) = E\{u|\theta\}$.

(c) Switch red and green and try (b) again.

Note: The function f is concave if $f(\alpha x + (1-\alpha)y) \geq \alpha f(x) + (1-\alpha)f(y)$. Pictorially, concave functions are $\cap$-shaped. Continuously differentiable concave function have second derivative ≤ 0. Jensen's inequality states that if g is a convex (concave) function, and θ is a random variable, than $E\{g(\theta)\} \geq\ (\leq)\ g(E\{\theta\})$.

4. (DeGroot, 1970). Suppose that Mr A and Mr B are equally wealthy, each having a current fortune of x_0 dollars. Suppose also that both men have the same utility function u and that for any number x, their utility of having a fortune of x dollars is $u(x) = (x - x_0)^{1/3}$. Suppose now that one of the two men receives, as a gift, a lottery ticket that yields either a reward of r dollars $(r \geq 0)$ or a reward of 0 dollars with equal probability. Show that there exist a number $b \geq 0$ having the following property.: regardless of which men receives the lottery ticket, he can sell it to the other man for b dollars and the sale will be advantageous for both men.