Homework Solutions 3

3.2 The simple events and corresponding Y values are

E_i	Y
HH	2
HT	-1
TH	-1
TT	1

Since $P(E_i) = \frac{1}{4}$ for each i, the probability distribution for Y is

y	-1	1	2
$p(y)$	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$

3.13 Let P be a random variable representing the company's profit. Then $P = C - 15$ with probability 98/100 (when A does not occur) and $P = C - 15 - 1000$ with probability 2/100 (when A occurs). Then $E(P) = (C-15)\frac{98}{100} + (C - 15 - 1000)\frac{2}{100} = 50$. Simplifying we have $C - 15 - 20 = 50$. Thus $C = \$85$.

3.16 Since the die is fair, the probability distribution for Y is

$$p(y) = \tfrac{1}{6} \qquad y = 1, 2, 3, 4, 5, 6$$

Then

$$E(Y) = \Sigma y p(y) = \tfrac{1}{6}(1+2+\ldots+6) = \tfrac{21}{6} = 3.5$$
$$E(Y^2) = \Sigma y^2 p(y) = \tfrac{1}{6}(1+4+9+\ldots+36) = \tfrac{91}{6} = 15.1667$$
$$V(Y) = E(Y^2) - [E(Y)]^2 = 15.1667 - (3.5)^2 = 2.9167$$

3.18 Consider first the probability distribution for y:

y	$p(y)$
0	.81
1	.18
2	.01

Then

$$\mu = E(Y) = \Sigma y p(y) = 0(.81) + 1(.18) + 2(.01) = .20$$

and

$$\sigma^2 = E(Y^2) - \mu^2 = [\Sigma y^2 p(y)] - \mu^2 = [0(.81) + 1(.18) + 4(.01)] - (.2)^2 = .22 - .04 = .18.$$

3.32 Let Y be the number of successful operations, with $n = 5$.

a. Use Table 1 with $p = .8$.
$p(5) = P(Y \le 5) - P(Y \le 4) = 1 - .672 = .328.$

b. For $p = .6$, $p(4) = P(Y \le 4) - P(Y \le 3) = .922 - .663 = .259.$

c. For $p = .3$, $P(Y < 2) = P(Y \le 1) = .528.$

3.42 The random variable Y is binomial with $n = 4$, $p = .1$. Hence

$$E(Y) = np = .4$$

and

$$E(Y^2) = V(Y) + [E(Y)]^2 = npq + n^2p^2 = 4(.1)(.9) + (.4)^2 = .52.$$

Then $E(C) = 3E(Y^2) + E(Y) + 2 = 3(.52) + .4 + 2 = 3.96.$

3.44 Let Y = # of fish that survive. Y is binomial with $n = 20$ and $p = 0.8$.

a. $P(Y = 14) = P(Y \le 14) - P(Y \le 13) = .196 - .087 = .109$

b. $P(Y \ge 10) = 1 - P(Y \le 9) = 1 - .001 = .999$

c. $P(Y \le 16) = .589$

d. $\mu = np = 20(.8) = 16$; $\sigma^2 = npq = 20(.8)(.2) = 3.2$

3.98 Let Y be the number of customers arriving. Then Y follows a Poisson distribution with $\lambda = 7$. We perform the calculations exactly, however one could just as easily used table 3 appendix III.

a. $P(Y \le 3) = p(0) + p(1) + p(2) + p(3) = \frac{7^0 e^{-7}}{0!} + \frac{7^1 e^{-7}}{1!} + \frac{7^2 e^{-7}}{2!} + \frac{7^3 e^{-7}}{3!} = .0818.$

b. $P(Y \ge 2) = 1 - P(Y \le 1) = 1 - \frac{7^0 e^{-7}}{0!} - \frac{7^1 e^{-7}}{1!} = 1 - 8e^{-7} = .9927.$

c. $P(Y = 5) = \frac{7^5 e^{-7}}{5!} = .1277.$

3.99 Let S = total service time = $10Y$. From 3.98 we know that $Y \sim$ Poisson(7). Therefore,

$$E(S) = 10E(Y) = 10(\lambda) = 10(7) = 70.$$
$$V(S) = (10)^2 V(Y) = 100\lambda = 100(7) = 700.$$

$$P(S > 150) = P(10Y > 150) = P(Y > 15) = 1 - P(Y \le 15)$$
$$= 1 - 0.998 = 0.002$$

where Table 3 was used to find $P(Y \le 15)$. So, we infer that it is unlikely that the total service time will exceed 2.5 hours.

3.106 The binomial probabilities for $n = 20$ and $p = .05$ are obtained by using the table as in previous exercises. However, for large n and small p such that $\lambda = np$ is less than 7, the following approximation can be used:

$$P(Y = r) \approx \frac{e^{-\lambda}\lambda^r}{r!} \qquad \text{where } \lambda = np \text{ and } r = 0, 1, 2, \dots, n$$

The exact binomial probabilities and their Poisson approximations are shown in the accompanying table. In this case, $\lambda = np = 20(.05) = 1$.

y	$p(y)$ (Exact Binomial)	$p(y)$ (Poisson Approximation)
0	.358	.368
1	.378	.368
2	.189	.184
3	.059	.061
4	.013	.015

Notice that the approximation is not too bad, even though n is fairly small.

4.6 **a.** The properties of a distribution function are satisfied since:

(1) $F(-\infty) = 0.$

(2) $F(\infty) = 1 - e^{-\infty} = 1 - 0 = 1.$

(3) $F(y_1) - F(y_2) = e^{-y_2^2} - e^{-y_1^2}$, which is positive if $y_1 > y_2$.

b. By Definition 4.3,

$$f(y) = F'(y) = \begin{cases} 2ye^{-y^2} & \text{for } y > 0 \\ 0 & \text{for } y \le 0 \end{cases}$$

c. $P(Y \ge 2) = 1 - P(Y < 2) = 1 - P(Y \le 2)$, since the probability at any particular point is 0. Thus,

$$P(Y \ge 2) = 1 - F(2) = 1 - (1 - e^{-4}) = e^{-4}.$$

d. $P(Y > 1 | Y \le 2) = \frac{P(1 < Y \le 2)}{P(Y \le 2)}$

So we need

$$P(1 < Y \le 2) = F(2) - F(1) = (1 - e^{-4}) - (1 - e^{-1}) = e^{-1} - e^{-4}.$$

Next we get $P(Y \le 2) = 1 - e^{-4}$ (using part **b.**). Thus

$$P(Y > 1 | Y \le 2) = \frac{(e^{-1} - e^{-4})}{1 - e^{-4}}.$$

4.12 **a.** $F(\infty) = \int_{-\infty}^{\infty} f(y)\,dy = \int_{-1}^{0} .2\,dy + \int_{0}^{1} (.2 + cy)\,dy = .2y\big]_{-1}^{0} + \left[.2y + \frac{cy^2}{2}\right]_0^1$

$= .2 + .2 + \frac{c}{2} = 1$ so that $c = 1.2$ and the density function is

$$f(y) = \begin{cases} .2, & -1 < y \le 0 \\ .2 + 1.2y, & 0 < y < 1 \\ 0, & \text{elsewhere} \end{cases}$$

b. $F(y) = 0$ for $y < -1$.

$$F(y) = \int_{-\infty}^{y} f(t)\,dy = \int_{-1}^{y} .2\,dt = .2t]_{-1}^{y} = .2y + .2 \qquad \text{for } -1 \le y \le 0$$

$$F(y) = \int_{-\infty}^{y} f(t)\,dy = \int_{-1}^{0} .2\,dt + \int_{0}^{y} (.2 + 1.2t)\,dt = .2 + [.2t + .6t^2]_0^y$$

$= .2 + .2y + .6y^2$ for $0 \le y \le 1$. $F(y) = 1$ for $y > 1$.

Collecting results, we have

$$F(y) = \begin{cases} 0, & y < -1 \\ .2(y+1), & -1 \le y \le 0 \\ .2(1 + y + 3y^2), & 0 < y \le 1 \\ 1, & y > 1 \end{cases}$$

c. The graphs of $f(y)$ and $F(y)$ are shown in Figures 4.8 and 4.9.

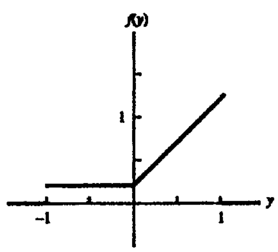

Figure 4.8

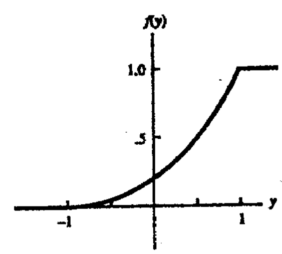

Figure 4.9

d. $F(-1) = .2(-1+1) = 0$, $F(0) = .2(0+1) = .2$, $F(1) = .2(1+1+3) = .2(5) = 1$

e. $P(0 \le Y \le .5) = F(.5) - F(0) = .2[1 + .5 + 3(.25)] - .2 = .2(2.25) - .2 = .25$

f. $P(Y > .5 | Y > .1) = \frac{P(Y>.5)}{P(Y>.1)} = \frac{1-.45}{1-.2(1+.1+.03)} = \frac{.55}{.774} = .71$

4.22 **a.** To solve for c, we need to consider $\int_0^1 cy^2(1-y)^4\,dy = 1$.

Integrating, we have $c\left(\frac{y^3}{3} - y^4 + \frac{6y^5}{5} - \frac{4y^6}{6} + \frac{y^7}{7}\right)\Big]_0^1 = 1$.

$c\left[\left(\frac{1}{3}\right) - 1 + \left(\frac{6}{5}\right) - \left(\frac{4}{6}\right) + \left(\frac{1}{7}\right)\right] = 1$.

$c = (1)\left(\frac{630}{6}\right) = 105$.

b. $E(Y) = 105\int_0^1 yy^2(1-y)^4\,dy = 105\left[\left(\frac{y^4}{4}\right) + \left(-\frac{4y^5}{5}\right) + y^6 + \left(-\frac{4y^7}{7} + \frac{y^8}{8}\right)\right]_0^1 = \frac{3}{8}$.