CHAPTER 5 MULTIVARIATE PROBABILITY DISTRIBUTIONS

5.1 Denote a sample space in terms of the firm that received the first and second contracts:

S	(y_1, y_2)
AA	$(2, 0)$
AB	$(1, 1)$
AC	$(1, 0)$
BA	$(1, 1)$
BB	$(0, 2)$
BC	$(0, 1)$
CA	$(1, 0)$
CB	$(0, 1)$
CC	$(0, 0)$

Each sample point is equally likely with probability $\frac{1}{9}$. Setting up a table for the joint function for Y_1 and Y_2,

		y_1		
		0	1	2
	0	$\frac{1}{9}$	$\frac{2}{9}$	$\frac{1}{9}$
y_2	1	$\frac{2}{9}$	$\frac{2}{9}$	0
	2	$\frac{1}{9}$	0	0

b. $F(1, 0) = P(Y_1 \le 1, Y_2 \le 0) = p(0, 0) + p(1, 0) = \frac{1}{9} + \frac{2}{9} = \frac{1}{3}$

5.2 The sample space for the toss of three balanced coins, the values for Y_1 and Y_2 at each outcome, and the probability of each outcome are given below:

OUTCOMES	(y_1, y_2)	PROBABILITY
HHH	$(3, 1)$	$\frac{1}{8}$
HHT	$(3, 1)$	$\frac{1}{8}$
HTH	$(2, 1)$	$\frac{1}{8}$
HTT	$(1, 1)$	$\frac{1}{8}$
THH	$(2, 2)$	$\frac{1}{8}$
THT	$(1, 2)$	$\frac{1}{8}$
TTH	$(1, 3)$	$\frac{1}{8}$
TTT	$(0, -1)$	$\frac{1}{8}$

		y_1			
		0	1	2	3
	-1	$\frac{1}{8}$	0	0	0
y_2	1	0	$\frac{1}{8}$	$\frac{2}{8}$	$\frac{1}{8}$
	2	0	$\frac{1}{8}$	$\frac{1}{8}$	0
	3	0	$\frac{1}{8}$	0	0

b. $F(2, 1) = P(Y_1 \le 2, Y_2 \le 1) = p(0, -1) + p(1, 1) + p(2, 1) = \frac{1}{8} + \frac{1}{8} + \frac{2}{8} = \frac{1}{2}$

5.3 In this exercise Y_1 and Y_2 are both discrete random variables, and the joint distribution for Y_1 and Y_2 is given by

$$P(Y_1 = y_1, Y_2, y_2) = p(y_1, y_2)$$

We must calculate $p(y_1, y_2)$ for $y_1 = 0, 1, 2, 3$ and $y_2 = 0, 1, 2, 3$. The total number of ways of choosing 3 persons for the committee is $\binom{9}{3} = 84$. Now,

$$P(Y_1 = 0, Y_2 = 0) = P(3 \text{ divorced}) = 0$$

since there are only 2 divorced executives available. However,

$$P(Y_1 = 1, Y_2 = 0) = P(1 \text{ married, } 0 \text{ never married, } 2 \text{ divorced}) = \frac{\binom{4}{1}\binom{3}{0}\binom{2}{2}}{\binom{9}{3}} = \frac{4}{84}.$$

Similar calculations, using an extension of the hypergeometric probability distribution discussed in Chapter 3, will allow one to obtain all 16 probabilities, and the joint probability distribution of Y_1 and Y_2 may be written in the form of a table.

$p(2,0) = \frac{\binom{4}{2}\binom{3}{0}\binom{2}{1}}{84} = \frac{12}{84}$ $\quad p(0,2) = \frac{\binom{4}{0}\binom{3}{2}\binom{2}{1}}{84} = \frac{6}{84}$

$p(1,1) = \frac{\binom{4}{1}\binom{3}{1}\binom{2}{1}}{84} = \frac{24}{84}$

$p(3,0) = \frac{\binom{4}{3}}{84} = \frac{4}{84}$ $\quad p(1,2) = \frac{\binom{4}{1}\binom{3}{2}}{84} = \frac{12}{84}$

$p(2,1) = \frac{\binom{4}{2}\binom{3}{1}\binom{2}{0}}{84} = \frac{18}{84}$

$p(0,3) = \frac{\binom{3}{3}}{84} = \frac{1}{84}$ $\quad p(0,1) = \frac{\binom{4}{0}\binom{3}{1}\binom{2}{2}}{84} = \frac{3}{84}$

$p(3,1) = p(2,2) = p(3,2) = p(3,3) = p(1,3) = p(2,3) = 0$

Note that $\sum_{y_1=0}^{3} \sum_{y_2=0}^{3} p(y_1, y_2) = 1$.

		y_2			
		0	1	2	3
y_1	0	0	$\frac{3}{84}$	$\frac{6}{84}$	$\frac{1}{84}$
	1	$\frac{4}{84}$	$\frac{24}{84}$	$\frac{12}{84}$	0
	2	$\frac{12}{84}$	$\frac{18}{84}$	0	0
	3	$\frac{4}{84}$	0	0	0

5.4 a. Notice that all of the probabilities are at least 0 and sum to 1

b. Note $F(1,2) = P(Y_1 \le 1, Y_2 \le 2) = 1$. The interpretation of this value is that every child in the experiment either survived or didn't and used either 0, 1 or 2 seatbelts.

5.6 a. We must have

$$F(\infty,\infty) = \int_0^1 \int_0^1 K y_1 y_2 \, dy_1 \, dy_2 = 1.$$

Then

$$\int_0^1 \int_0^1 K y_1 y_2 \, dy_1 \, dy_2 = K \int_0^1 (y_2) \left[\frac{y_1^2}{2}\right]_0^1 dy_2 = \frac{K}{2} \int_0^1 y_2 \, dy_2 = \frac{K}{2} \left[\frac{y_2^2}{2}\right]_0^1 = \frac{K}{4} = 1$$

so that $K = 4$.

b. $F(y_1, y_2) = \int_0^{y_2} \int_0^{y_1} 4t_1 t_2 \, dt_1 \, dt_2 = \int_0^{y_2} \left[\frac{4t_1^2}{2}\right]_0^{y_1} dt_2 = \int_0^{y_2} 2y_1^2 t_2 \, dt_2 = y_1^2 y_2^2$

for $0 \le y_1 \le 1$ and $0 \le y_2 \le 1$. Recall that

$$F(y_1, y_2) = \begin{cases} 0, & \text{for } y_1 \le 0 \text{ or } y_2 \le 0 \\ 1, & \text{for } y_1 \ge 1 \text{ and } y_2 \ge 1. \end{cases}$$

c. $P\left(Y_1 \le \frac{1}{2}, Y_2 \le \frac{3}{4}\right) = F\left(\frac{1}{2}, \frac{3}{4}\right) = \left(\frac{1}{2}\right)^2 \left(\frac{3}{4}\right)^2 = \frac{9}{64}$

5.14 a. $P\left(Y_1 < \frac{1}{2}, Y_2 > \frac{1}{4}\right) = \int_{1/4}^{1} \int_0^{1/2} (y_1 + y_2) \, dy_1 \, dy_2 = \int_{1/4}^{1} \left(\frac{1}{8} + \frac{y_2}{2}\right) dy_2 = \frac{21}{64}$

5.17 a.

y_1	0	1	2
$p(y_1)$	$\frac{4}{9}$	$\frac{4}{9}$	$\frac{1}{9}$

b. No. Evaluating $f(y) = \binom{2}{y}\left(\frac{1}{3}\right)^y \left(\frac{2}{3}\right)^{2-y}$ for each value of Y_1 will result in the same probabilities as those given in part **a**.

5.18 a.

y_2	-1	1	2	3
$p(y_2)$	$\frac{1}{8}$	$\frac{4}{8}$	$\frac{2}{8}$	$\frac{1}{8}$

b. $P(Y_1 = 3 | Y_2 = 1) = \frac{P(Y_1=3, Y_2=1)}{P(Y_2=1)} = \frac{\left(\frac{1}{8}\right)}{\left(\frac{4}{8}\right)} = \frac{1}{4}$

5.20 a. The marginal distributions for Y_1 and Y_2 are given in the margins of the table. That is, the marginal distribution for Y_1 is $P(Y_1 = 0) = .76$ and $P(Y_1 = 1) = .24$ and the marginal distribution for Y_2 is given by $P(Y_2 = 0) = .55$, $P(Y_2 = 1) = .16$ and $P(Y_2 = 2) = .29$.

b. $P(Y_2 = 0|Y_1 = 0) = \frac{P(Y_2=0, Y_1=0)}{P(Y_1=0)} = \frac{.38}{.76} = .5$, $P(Y_2 = 1|Y_1 = 0) = \frac{.14}{.76} = .18$
$P(Y_2 = 2|Y_1 = 0) = \frac{.24}{.76} = .32$.

c. The desired probability $P(Y_1 = 0|Y_2 = 2) = \frac{.38}{.55} = .69$.

5.22 a. By definition,

$$f_1(y_1) = \int_{-\infty}^{\infty} f(y_1, y_2)\, dy_2 = \int_0^1 4y_1y_2\, dy_2 = (4y_1)\left(\frac{y_2^2}{2}\right)\Big]_0^1 = 2y_1 \quad \text{for } 0 \le y_1 \le 1$$

and

$$f_2(y_2) = \int_0^1 4y_1y_2\, dy_1 = (4y_2)\left(\frac{y_1^2}{2}\right)\Big]_0^1 = 2y_2 \qquad \text{for } 0 \le y_2 \le 1$$

b. By the definition of conditional probability,

$$P\left(Y_1 \le \tfrac{1}{2}|Y_2 > \tfrac{3}{4}\right) = \frac{P\left(Y_1 \le \frac{1}{2}, Y_2 > \frac{3}{4}\right)}{P\left(Y_2 > \frac{3}{4}\right)}.$$

Now

$$P\left(Y_1 \le \tfrac{1}{2}, Y_2 > \tfrac{3}{4}\right) = \int_0^{1/2} \int_{3/4}^{1} 4y_1y_2\, dy_2\, dy_1 = \int_0^{1/2} 2y_1 \left[y_2^2\right]_{3/4}^1 dy_1 = \tfrac{7}{16} y_1^2\Big|_0^{1/2} = \tfrac{7}{64}$$

and

$$P\left(Y_2 > \tfrac{3}{4}\right) = \int_{3/4}^{1} f_2(y_2)\, dy_2 = \int_{3/4}^{1} 2y_2\, dy_2 = y_2^2\Big]_{3/4}^1 = \tfrac{7}{16}.$$

Hence

$$P\left(Y_1 \le \tfrac{1}{2}|Y_2 > \tfrac{3}{4}\right) = \frac{\left(\frac{7}{64}\right)}{\left(\frac{7}{16}\right)} = \tfrac{1}{4}.$$

Notice this the same probability as $P(Y_1 \le \frac{1}{2})$.

c. By Definition 5.7, if $0 < y_2 \le 1$

$$f(y_1|y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{4y_1y_2}{2y_2} = 2y_1, \qquad 0 \le y_1 \le 1.$$

Notice this is the same as $f(y_1)$.

d. If $0 < y_1 \le 1$,

$$f(y_2|y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} = \frac{4y_1y_2}{2y_1} = 2y_2, \qquad 0 \le y_2 \le 1.$$

Notice this is the same as $f(y_2)$

e. $P\left(Y_1 \le \frac{3}{4}|Y_2 = \frac{1}{2}\right) = \int_0^{3/4} f\left(y_1|y_2 = \frac{1}{2}\right) dy_2 = \int_0^{3/4} 2y_1\, dy_1 = y_1^2\Big]_0^{3/4} = \frac{9}{16}$

5.30 a.

$$f_1(y_1) = \int_0^1 (y_1 + y_2)\, dy_2 = y_1 + \tfrac{1}{2} \qquad 0 \le y_1 \le 1$$

$$f_2(y_2) = \int_0^1 (y_1 + y_2)\, dy_1 = y_2 + \tfrac{1}{2} \qquad 0 \le y_2 \le 1$$

b. Calculate

$$P\left(Y_2 \ge \tfrac{1}{2}\right) = \int_{1/2}^{1} \left(y_2 + \tfrac{1}{2}\right) dy_2 = \left[\tfrac{1}{2} y_2 + \tfrac{y_2^2}{2}\right]_{1/2}^{1} = \tfrac{5}{8}$$

$$P\left(Y_1 \ge \tfrac{1}{2}, Y_2 \ge \tfrac{1}{2}\right) = \int_{1/2}^{1} \int_{1/2}^{1} (y_1 + y_2)\, dy_1\, dy_2 = \int_{1/2}^{1} \left(\tfrac{3}{8} + \tfrac{y_2}{2}\right) dy_2 = \tfrac{3}{8}$$

Hence

$$P\left(Y_1 \ge \tfrac{1}{2}|Y_2 \ge \tfrac{1}{2}\right) = \frac{\left(\frac{6}{16}\right)}{\left(\frac{5}{8}\right)} = \tfrac{3}{5}$$

c. First consider $f(y_1|y_2) = \frac{f(y_1, y_2)}{f(y_2)}$. If $0 \le y_2 \le 1$ we have

$$f(y_1|y_2) = \frac{y_1 + y_2}{y_2 + \frac{1}{2}} \qquad 0 \le y_1 \le 1$$

Then

$$
\begin{aligned}
P(Y_1 > .75|Y_2 = .5) &= \int_{.75}^{1} \frac{y_1 + \frac{1}{2}}{\frac{1}{2} + \frac{1}{2}}\, dy_1 \\
&= \left(\frac{1}{2}\right) y_1^2 + \left(\frac{1}{2}\right) y_1 \Big]_{.75}^{1} \\
&= \left(\frac{1}{2}\right) + \left(\frac{1}{2}\right) - .28125 - .375 = .34375 \\
&= .34375
\end{aligned}
$$

5.39 No. For example, consider $P(Y_1 = 0, Y_2 = 0)$ and $p(Y_1 = 0)p(Y_2 = 0)$

$$p(0, 0) = \tfrac{1}{9} \neq \left(\tfrac{4}{9}\right)\left(\tfrac{4}{9}\right) = p_1(0)p_2(0).$$

Thus, Y_1 and Y_2 are not independent.

5.40 No. Considering $P(Y_1 = 3, Y_2 = 1)$ and $p(Y_1 = 3)p(Y_2 = 1)$

$$p(3, 1) = \tfrac{1}{8} \neq \left(\tfrac{1}{8}\right)\left(\tfrac{4}{8}\right) = p_1(3)p_2(1)$$

Thus, Y_1 and Y_2 are not independent.

5.42 Dependent, for example $P(Y_1 = 0, Y_2 = 0) \neq P(Y_1 = 0)P(Y_2 = 0)$.

5.44 Independent as $f(y_1, y_2)$ can be factored (Theorem 5.5).

5.62
a. $E(Y_1) = np = 2\left(\frac{1}{3}\right) = \frac{2}{3}$.
b. $V(Y_1) = np(1 - p) = 2\left(\frac{1}{3}\right)\left(\frac{2}{3}\right) = \frac{4}{9}$.
c. $E(Y_1 - Y_2) = E(Y_1)\_ - E(Y_2) = \left(\frac{2}{3}\right) - \left(\frac{2}{3}\right) = 0$.

5.64 Refer to Exercises 5.6 and 5.22. Recall $f_1(y_1) = 2y_1$ for $0 \leq y_1 \leq 1$.
a. $E(Y_1) = \int_0^1 2y_1 y_1\, dy_1 = \int_0^1 2y_1^2\, dy_1 = \frac{2}{3}$
b. $E(Y_1^2) = \int_0^1 2y_1^3\, dy_1 = \frac{1}{2}$ so that $V(Y_1) = \frac{1}{2} - \frac{4}{9} = \frac{1}{18}$.
c. Since $E(Y_2) = \int_0^1 2y_2^2\, dy_2 = \frac{2}{3}$, $E(Y_1 - Y_2) = 0$.

5.68 Refer to Exercise 5.14.

$$
\begin{aligned}
E(Y_1) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1 f(y_1, y_2)\, dy_1\, dy_2 = \int_0^1\int_0^1 y_1(y_1 + y_2)\, dy_1\, dy_2 \\
&= \int_0^1 \left[\frac{y_1^3}{3} + \frac{y_1^2 y_2}{2}\right]_0^1 dy_2 = \left[\frac{1}{3} y_2 + \frac{y_2^2}{4}\right]_0^1 = \frac{7}{12}
\end{aligned}
$$

By symmetry, $E(Y_2) = \frac{7}{12}$ and $E(30Y_1 + 25Y_2) = (30 + 25)\left(\frac{7}{12}\right) = 32.08$.

5.75 $\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2)$.

$$
\begin{aligned}
E(Y_1Y_2) &= \sum_{y_1}\sum_{y_2} y_1 y_2 p(y_1, y_2) = (0)(0)\left(\tfrac{1}{9}\right) + (1)(0)\left(\tfrac{2}{9}\right) + (2)(0)\left(\tfrac{1}{9}\right) + (0)(1)\left(\tfrac{2}{9}\right) \\
&\quad + (1)(1)\left(\tfrac{2}{9}\right) + (0)(2)\left(\tfrac{1}{9}\right) = \tfrac{2}{9}.
\end{aligned}
$$

Since Y_1 and Y_2 are both binomial with $n = 2$ and $p = \frac{1}{3}$,

$$E(Y_1) = E(Y_2) = 2\left(\tfrac{1}{3}\right) = \tfrac{2}{3}.$$

Thus, $\text{Cov}(Y_1, Y_2) = \left(\frac{2}{9}\right) - \left(\frac{2}{3}\right)\left(\frac{2}{3}\right) = -\frac{2}{9}$.

No, as value of Y_1 increases, value of Y_2 tends to decrease.

5.77 From Exercise 5.46; $E(Y_1) = E(Y_2) = \frac{2}{3}$. Then

$$E(Y_1Y_2) = \int_0^1 \int_0^1 4y_1^2y_2^2\, dy_1\, dy_2 = \int_0^1 \tfrac{4}{3}\, y_2^2\, dy_2 = \tfrac{4}{9}$$

$$\text{Cov}(Y_1, Y_2) = \tfrac{4}{9} - \tfrac{4}{9} = 0.$$

No, this is not surprising since Y_1 and Y_2 are independent.

5.80 $\text{Cov}(U_1, U_2) = E\{(Y_1 + Y_2)(Y_1 - Y_2 - [E(Y_1) + E(Y_2)][E(Y_1) - E(Y_2)]\}$

$$= E(Y_1Y_2) + E(Y_1^2) - E(Y_1Y_2) - E(Y_2^2) - [E(Y_1)]^2 - E(Y_1)E(Y_2) + E(Y_1)E(Y_2) + [E(Y_2)]^2$$

$$= \sigma_1^2 - \sigma_2^2$$

Now

$$V(U_1) = E[U_1^2] - [E(U_1)]^2$$
$$= E(Y_1^2 + 2Y_1Y_2 + Y_2^2) - [(EY_1)^2 + 2(EY_1)(EY_2) + (EY_2)^2]$$
$$= V(Y_1) + V(Y_2) + 2[E(Y_1Y_2) - (EY_1)EY_2)]$$
$$= \sigma_1^2 + \sigma_2^2 + 2\text{Cov}(Y_1, Y_2)$$
$$= \sigma_1^2 + \sigma_2^2$$

since Y_1 and Y_2 are uncorrelated. A similar calculation yields $V(U_2) = \sigma_1^2 + \sigma_2^2$. Hence

$$\rho = \frac{\sigma_1^2 - \sigma_2^2}{\sqrt{(\sigma_1^2 + \sigma_2^2)(\sigma_1^2 + \sigma_2^2)}} = \frac{\sigma_1^2 - \sigma_2^2}{\sigma_1^2 + \sigma_2^2}$$

5.86 Let X = dollar amount spent per week $= 3Y_1 + 5Y_2$.

$$E(X) = E[3Y_1 + EY_2] = 3E(Y_1) + 5E(Y_2) = 3(40) + 5(65) = 445.$$

$$V(X) = V[3Y_1 + 5Y_2] = 9V(Y_1) + 25V(Y_2) \text{ since } Y_1 \text{ and } Y_2 \text{ are independent}$$
$$= 9(4) + 25(8) = 236.$$

5.89 $V(Y_1 - Y_2) = \frac{1}{18} + \frac{1}{18} - 2(0) = \frac{1}{9}$

(See Exercise 5.64 for $V(Y_1)$. Also, $V(Y_2) = V(Y_1)$ by symmetry.)

5.93 Several intermediate results will be necessary.

(i) From Exercise 5.50, $E(Y_1) = \frac{7}{12}$ and $E(Y_2) = \frac{7}{12}$.

(ii) $E(Y_1Y_2) = \int_0^1 \int_0^1 (y_1 + y_2)\, y_1y_2\, dy_1\, dy_2 = \int_0^1 \left[\frac{y_1^3y_2}{3} + \frac{y_1^2y_2^2}{2}\right]_0^1 dy_2$

$$= \int_0^1 \left(\frac{y_2}{3} + \frac{y_2^2}{2}\right) dy_2 = \left[\frac{y_2^2}{6} + \frac{y_2^3}{6}\right]_0^1 = \frac{1}{3}$$

(iii) $V(Y_1) = \int_0^1 \int_0^1 (y_1^3 + y_1^2y_2)\, dy_2\, dy_1 - [E(Y_1)]^2 = \int_0^1 \left(y_1^3 + \frac{1}{2}y_1^2\right) dy_1 - \frac{49}{144}$

$$= \left[\frac{y_1^4}{4} + \frac{y_1^3}{6}\right]_0^1 - \frac{49}{144} = \frac{11}{144}$$

and $V(Y_2) = V(Y_1) = \frac{11}{144}$.

(iv) $\text{Cov}(Y_1, Y_2) = E(Y_1Y_2) - E(Y_1)E(Y_2) = \frac{1}{3} - \left(\frac{7}{12}\right)\left(\frac{7}{12}\right) = -\frac{1}{144} = .0069$

Thus

$$E(30Y_1 + 25Y_2) = 30E(Y_1) + 25E(Y_2) = 32.08$$

and

$$V(30Y_1 + 25Y_2) = 900\left(\tfrac{11}{144}\right) + 625\left(\tfrac{11}{144}\right) + 2(750)\left(-\tfrac{1}{144}\right) = 106.08.$$

Then $\sigma = \sqrt{V(30Y_1 + 25Y_2)} = 10.30$.

Using Tchebysheff's theorem with $k = 2$, the necessary interval is $\mu \pm 2\sigma = 32.08 \pm 2(10.30) = 32.08 \pm 20.6$, or 11.48 to 52.68