9.72 a. The likelihood function is

$$L = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y!} = \frac{\lambda^{\Sigma y_i} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}$$

and

$$\ln L = (\Sigma y_i) \ln \lambda - n\lambda - \Sigma \ln y_i!.$$

so that $\left(\frac{d}{d\lambda}\right)[\ln L] = \left(\Sigma \frac{y_i}{\lambda}\right) - n$. Equating the derivative to 0, we obtain

$$\frac{\Sigma y_i}{\hat{\lambda}} - n = 0$$

or

$$\hat{\lambda} = \frac{\Sigma Y_i}{n} = \overline{Y}.$$

b. Recalling that $E(Y_i) = \lambda$ and $V(Y_i) = \lambda$, we obtain

$$E(\hat{\lambda}) = \frac{\sum_{i=1}^{n} E(Y_i)}{n} = \lambda$$

and

$$V(\hat{\lambda}) = \frac{\sum_{i=1}^{n} V(Y_i)}{n^2} = \frac{\lambda}{n}.$$

c. Since $E(Y_i) = \lambda$ and $V(Y_i) = \lambda < \infty$, the law of large numbers applies and we conclude that $\hat{\lambda}$ converges in probability to λ. Hence $\hat{\lambda}$ is consistent for λ.

d. The MLE of λ was found in part a to be $\hat{\lambda} = \overline{Y}$. Then, the MLE for $e^{-\lambda}$ is $e^{-\overline{Y}}$.

9.74 a. The likelihood function is

$$L = \prod_{i=1}^{n} \frac{r}{\theta} y_i^{r-1} e^{-y_i^r/\theta} = \frac{r^n}{\theta^n} \prod_{i=1}^{n} y^{r-1} e^{-\Sigma y_i^r/\theta} = g(u, \theta) h(y_1, y_2, \ldots, y_n)$$

where

$$u = \sum_{i=1}^{n} y_i^r \qquad g(u, \theta) = \frac{r^n}{\theta^n} e^{-u/\theta} \qquad h(y_1, y_2, \ldots, y_n) = \prod_{i=1}^{n} y_i^{r-1}$$

Hence ΣY_i^r is a sufficient statistic for θ.

b. Consider $\ln L = n \ln r - n \ln \theta + (r-1) \sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n} \frac{y_i^r}{\theta}$ and $\frac{d}{d\theta} \ln L = \frac{-n}{\theta} + \frac{\Sigma y_i^r}{\theta^2}$.

Equating the derivative to 0, the estimator is obtained.

$$\frac{-n}{\hat{\theta}} + \frac{\Sigma y_i^r}{\hat{\theta}^2} = 0 \qquad \text{or} \qquad -n\hat{\theta} + \Sigma y_i^r = 0 \qquad \text{or} \qquad \hat{\theta} = \frac{\Sigma Y_i^r}{n}.$$

9.76 a. As this exercise is a special case of exercise 9.77 **a** (with $\alpha = 2$) we will refer to its results.

$$\hat{\theta} = \left(\frac{\overline{Y}}{2}\right) = \frac{378}{3(2)} = 63.$$

b. From Exercise 9.69 **b**,

$$E(\hat{\theta}) = \theta \qquad V(\hat{\theta}) = \frac{\theta^2}{n\alpha} = \frac{\theta^2}{3(2)} = \frac{\theta^2}{6}$$

c. The bound on the error of estimation is

$$2\sqrt{V(\hat{\theta})} = 2\sqrt{\frac{\theta^2}{6}} = 2\sqrt{\frac{(130)^2}{6}} = 106.14$$

d. The variance of Y is $2\theta^2$. The MLE of θ was found in part **a** to be $\hat{\theta} = 63$. Therefore, the MLE for the variance is $2(63)^2 = 7938$.

9.78 The likelihood function is

$$L = \prod_{i=1}^{m} \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{x_i-\mu_1}{\sigma}\right)^2\right] \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\theta} \exp\left[-\frac{1}{2}\left(\frac{y_i-\mu_2}{\sigma}\right)^2\right]$$

$$= \frac{1}{(2\pi)^{(m+n)/2}\sigma^{m+n}} \exp\left\{-\frac{1}{2}\left[\sum_{i=1}^{m}\left(\frac{x_i-\mu_1}{\sigma}\right)^2 + \sum_{i=1}^{n}\left(\frac{y_i-\mu_2}{\sigma}\right)^2\right]\right\}$$

and

$$\ln L = \ln K - (m+n)\ln\sigma - \frac{1}{2\sigma^2}\left[\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{i=1}^{n}(y_i-\mu_2)^2\right]$$

Then

$$\frac{d}{d\sigma}\ln L = \frac{-(m+n)}{\sigma} + \frac{1}{\sigma^3}\left[\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{i=1}^{n}(y_i-\mu_2)^2\right]$$

Setting the derivative equal to 0 and solving for $\hat{\sigma}$, we have

$$\frac{m+n}{\hat{\sigma}} = \frac{1}{\hat{\sigma}^3}\left[\sum_{i=1}^{m}(x_i-\mu_1)^2 + \sum_{i=1}^{n}(y_i-\mu_2)^2\right]$$

or

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{m}(X_i-\mu_1)^2 + \sum_{i=1}^{n}(Y_i-\mu_2)^2}{m+n}.$$

Since μ_1 and μ_2 are unknown, their maximum likelihood estimates must be obtained.

$$\frac{d}{d\mu_1}\ln L = \frac{\sum_{i=1}^{m}(x_i-\mu_1)}{\sigma^2} \quad \text{and} \quad \frac{d}{d\mu_2}\ln L = \frac{\sum_{i=1}^{n}(y_i-\mu_2)}{\sigma^2}$$

and, as in Example 9.15 in the text, $\hat{\mu}_1 = \overline{X}$ and $\hat{\mu}_2 = \overline{Y}$. Thus,

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{m}(X_i-\overline{X})^2 + \sum_{i=1}^{n}(Y_i-\overline{Y})^2}{m+n}$$

9.81 $P(Y=y) = \binom{2}{y} p^y(1-p)^{2-y}$. Our estimator, $\hat{p}$, must be either 1/4 or 3/4. We choose based on which has the larger likelihood value given the data, Y. It is important to remember in this problem that the likelihood is a function of the parameter p. Therefore we have three possible likelihood functions depending, one for each value of the data, Y.

$L(0,p) = P(Y=0) = (1-p)^2$ implying $\hat{p} = \frac{1}{4}$ as
$L(0,\frac{1}{4}) = (1-\frac{1}{4})^2 > (1-\frac{3}{4})^2 = L(0,\frac{3}{4})$.
$L(1,p) = P(Y=1) = 2p(1-p)$ implying $\hat{p}$ can be either $\frac{1}{4}$ or $\frac{3}{4}$ as
$L(1,\frac{1}{4}) = 2\frac{1}{4}(1-\frac{1}{4}) = 2\frac{3}{4}(1-\frac{3}{4}) = L(1,\frac{3}{4})$.
$L(2,p) = P(Y=2) = p^2$ implying $\hat{p} = \frac{3}{4}$ as $(\frac{1}{4})^2 < (\frac{3}{4})^2$.

Notice the case when $Y=1$ is an instance where the maximum likelihood estimator is not a single unique value!

9.82 Notice under the hypothesis $p_W = p_M = p$ the number of people of our sample who favor the issue is binomial with success probability p and number of trials equal to 200. Then by problem 9.14 we have $\hat{p} = \frac{55}{200}$.

9.83 [illegible]