Probability

What Is Probability?

- Many events are not certain. These are called random events.

  Example: Price of a particular stock on next Monday, blood pressure of a person tomorrow at 2:00pm *etc.*

- The probability of a random event is a measure of likeliness that the event will occur.

- If an experiment is repeated independently and under the same conditions, then the stable long-term relative frequency of a random event is called its probability.

- Example: If a coin is balanced or unbiased, then in a long series of tosses there will be about 50% heads, although one can not predict what will happen in a particular toss.

Review of Set Notations

- Union of two sets A and B: $A \cup B$

  (means A or B or both)

- Intersection of A and B: $A \cap B$

  (means A and B)

- Complement of a set A: A^c or $\overline{A}$

  (means not A)

  (Note that the book uses $\overline{A}$ instead of A^c.)

Event, Simple Event, Sample Space and Sample Point (Discrete Case)

- A *simple event* is an event that can not be decomposed. It is a set consisting of exactly one element.

- The element of a simple event is called a *sample point.*

- The *sample space* (denoted by S) is the set of all sample points. A discrete sample space has either finite or countable number of sample points.

- An *event* is a subset of a sample space.

- Example: A die is rolled. Let

  $A =$ The number on the upper side is even.

  $B =$ The number on the upper side is 1.

  $C =$ The number on the upper side is 2.

  The sample space is $\{e_1, e_2, e_3, e_4, e_5, e_6\}$, where e_i corresponds to number i appearing on the the upper side of the die. A, B, C are all events, e_i's are sample points. B and C are simple events, A is not.

Probability Axioms

- $P(A) \geq 0$ for any event A.

- $P(S) = 1$.

- If $A_1, A_2, \ldots$ are pairwise mutually disjoint (i.e., $A_i \cap A_j = \Phi$ for $i \neq j$), then

$$P(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i).$$

  Here $\cup_{i=1}^{\infty} A_i = A_1 \cup A_2 \cup \cdots$.

  For finite number of pairwise mutually disjoint events $A_1, A_2, \ldots, A_n$,

$$P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i).$$

Some Rules Deducible from the Above Axioms

1. If $A \subset B$, then $P(A) \leq P(B)$.

2. $P(\overline{A}) = 1 - P(A)$.

3. $P(\cup_{i=1}^{n} A_i) = \sum_{i=1}^{n} P(A_i)$

   $-\sum_{i=1}^{n} \sum_{j=i+1}^{n} P(A_i \cap A_j) + \cdots + (-1)^{n+1} P(\cap_{i=1}^{n} A_i)$.

   Two special cases are:

   (a) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

   (b) $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3)$

   $- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3)$

   $+ P(A_1 \cap A_2 \cap A_3)$.

Calculating Probabilities by Sample Point Method

- Define the experiment.

- List the simple events and make sure that they can not be decomposed. Write the sample space consisting of the sample points corresponding to the simple events.

- Assign probability to each simple event E_i such that $P(E_i) \geq 0$ and $\sum_i P(E_i) = 1$.

- Decompose an event into simple events:

$$A = \cup_{i \in I} E_i.$$

  Here I is an index set.

- Sum the probabilities of the distinct simple events in A:

$$P(A) = \sum_{i \in I} P(E_i).$$

Some Combinatorial Results

1. The mn rule

   - The number of pairs that can be formed by choosing one element form each of two groups containing m and n elements, respectively, is mn.

   Proof. Group 1 $= \{a_1, a_2, \ldots, a_m\}$ and Group 2 $= \{b_1, b_2, \ldots, b_n\}$. There are m ways to choose an element from Group 1. For each of m ways, there are n ways to choose an element from Group 2. So, there are total mn ways to choose one element from each group.

- This rule can be generalized to more than two groups.

2. <u>Permutations</u>

- A permutation is an ordered arrangement of distinct objects.

- The number of permutations of n distinct objects taken r at a time is denoted by P_r^n.

- $P_r^n = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$

Proof. P_r^n is the number of of ways r ordered positions can be filled by choosing r objects from a group of n distinct objects. The first position can be filled in n ways. For each of these n ways, the second position can be filled in $n-1$ ways. So, the first two positions can be filled in $n(n-1)$ ways. For each of these $n(n-1)$ ways, the third position can be filled in $n-2$ ways and so on. After the r-th step, we get the result.

- Special case: $P_n^n = n!$

3. Combinations

- A combination is a collection of objects.
- The number of combinations of n objects taken r at a time is denoted by C_r^n or by $\binom{n}{r}$.
- $\binom{n}{r} = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}$

Proof. One can obtain a permutation in two steps, first choosing r objects from n objects and then arranging r objects. From n objects, r objects can be chosen chosen in $\binom{n}{r}$ ways. For each of these ways the chosen r objects can be arranged in P_r^r ways. This gives all P_r^n permutations. So, $P_r^n = \binom{n}{r} P_r^r$. Therefore, $\binom{n}{r} = \frac{P_r^n}{P_r^r} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$.

4. <u>Partitioning into k groups</u>

- The number of ways of partitioning n distinct objects into k distinct groups containing n_1, n_2, ..., n_k objects, respectively, where $\sum_{i=1}^{k} n_i = n$, is $\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1! n_2! \cdots n_k!}$

Proof. This can be done in k steps. The number of ways of choosing n_1 objects from n objects is $\binom{n}{n_1}$. For each of these ways, one can choose n_2 objects from the remaining $n - n_1$ objects in $\binom{n-n_1}{n_2}$ ways. So, the number of ways of forming two groups containing n_1 and n_2 objects is $\binom{n}{n_1}\binom{n-n_1}{n_2}$. Proceeding in this way we can show that

$$\begin{aligned}
&\binom{n}{n_1, n_2, \ldots, n_k} \\
=&\binom{n}{n_1}\binom{n-n_1}{n_2}\binom{n-n_1-n2}{n_3}\cdots\binom{n_{k-1}+n_k}{n_{k-1}}\binom{n_k}{n_k} \\
=&\frac{n!}{n_1!n_2!\cdots n_k!}
\end{aligned}$$

Finding the Probabilities with Equiprobable Simple Events

- If each simple events has the same probability, one can just count the number of sample points in the sample space and in an event to find the probability of that event.

- If an event A has n_A sample points and the sample space has N sample points, then $P(A) = \frac{n_A}{N}$.

- The combinatorial results will be useful in calculating the probability of an event when the simple events are equiprobable.

Conditional Probability

- The conditional probability of A given B (denoted by $P(A \mid B)$) is the probability of A when B is known.

- $P(A \mid B) = \frac{P(A \cap B)}{P(B)}$ if $P(B) > 0$.

- Example: A die is rolled.

  P(observing a 2 on the upper side) $= \frac{1}{6}$.

  P(observing a 2 on the upper side | the upper side shows an even number) $= \frac{1}{3}$.

Independence

- Two events are independent if the probability of one is not affected by occurrence or non-occurrence of the other.

- To check whether A and B are independent, check if one of the following equivalent conditions is satisfied.

  $$P(A \cap B) = P(A) \cdot P(B)$$

  $$P(A \mid B) = P(A)$$

  $$P(B \mid A) = P(B)$$

Multiplicative and Additive Laws

- Multiplicative Law: $P(A \cap B) = P(A) \cdot P(B \mid A)$

$$= P(B) \cdot P(A \mid B)$$

- Additive Law (we have already seen it):

$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$.

Calculating Probabilities by Event Composition Method

- There is no general rule, need to practice.
- Use the multiplicative and additive laws and $P(\overline{A}) = 1 - P(A)$.
- One particular technique is the use of the law of total probability.

The Law of Total Probability and Bayes' Rule

- Definition: A collection of sets $\{B_1, \ldots, B_k\}$ is called a partition of S if

  1. $S = B_1 \cup B_2 \cup \cdots \cup B_k$ and
  2. $B_i \cap B_j = \Phi$ for $i \neq j$.

- The law of total probability: If $\{B_1, \ldots, B_k\}$ is a partition of S with $P(B_i) > 0$ for $i = 1, 2, \ldots, k$, then for any event A from the sample space S, we have $P(A) = \sum_{i=1}^{k} P(A \mid B_i) \cdot P(B_i)$.

- Bayes' rule: If $\{B_1, \ldots, B_k\}$ is a partition of S with $P(B_i) > 0$ for $i = 1, 2, \ldots, k$, and A is any event from the sample space S with $P(A) > 0$, then

$$P(B_j \mid A) = \frac{P(A \mid B_j) \cdot P(B_j)}{\sum_{i=1}^{k} P(A \mid B_i) \cdot P(B_i)}.$$