Solutions for quiz 9

1. (a)

$$L(\theta) = f(x_1, x_2, ...x_n|\theta) = \prod_{i=1}^{n} f(x_i|\theta)$$

$$= \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} = \theta^n(1-\theta)^{\sum_{i=1}^{n}(x_i-1)}$$

$$\ln L(\theta) = n\ln\theta + (\sum_{i=1}^{n} x_i - n)\ln(1-\theta)$$

$$\frac{\partial \ln L(\theta)}{\partial\theta} = \frac{n}{\theta} - \frac{(\sum_{i=1}^{n} x_i - n)}{1-\theta} = 0$$

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} x_i} = \frac{1}{\bar{X}}$$

$$\frac{\partial^2 \ln L(\theta)}{\partial\theta^2} = -\frac{n}{\theta^2} - \frac{(\sum_{i=1}^{n} x_i - n)}{(1-\theta)^2}$$

$$\begin{aligned}
\frac{\partial^2 \ln L(\theta)}{\partial\theta^2}\Big|_{\theta=\hat{\theta}} &= -\frac{n}{\frac{1}{\bar{X}^2}} - \frac{n\bar{X}-n}{(1-\frac{1}{\bar{X}})^2} \\
&= -n\bar{X}^2 - \frac{n\bar{X}^2}{\bar{X}-1} \\
&= -n\bar{X}^2(1+\frac{1}{\bar{X}-1}) \\
&= -n\bar{X}^2\frac{\bar{X}}{\bar{X}-1} \\
&= \frac{-n\bar{X}^3}{\bar{X}-1} \\
&< 0 \qquad \text{unless all } X_i\text{s are 1 since } X_i \geq 1 \text{ for each } i.
\end{aligned}$$

If all X_is are 1, then $L(\theta) = \theta^n$ and it is maximum when $\theta = 1$ because θ^n is increasing in θ for $\theta \geq 0$ and the maximum possible value of θ is 1.

(b) By the invariance property of MLE,

$$\hat{\text{Var}}(\theta) = \frac{1-\hat{\theta}}{\hat{\theta}^2} = \frac{1-\frac{1}{\bar{X}}}{(\frac{1}{\bar{X}})^2} = \bar{X}(\bar{X}-1)$$

2. The confidence interval for the mean has the form

$$\bar{Y} \pm t_{\alpha/2}\frac{S}{\sqrt{n}}$$

and the t distribution has n-1 degrees of freedom.
Here $\bar{Y} = 8.5$, $S = .08$, $n = 9$ and $t_{.05}$ with 8 degree of freedom is 1.86.
Therefore, the 90% confidence interval is

$$8.5 \pm 1.86(\frac{.08}{3}) = [8.4504, 8.5496]$$