STA205 Probability & Measure

Homework #1

Due 2002 April 3

1. For real-valued random variables X, Y on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, define
$$d(X,Y) = \mathsf{E}\left[\frac{|X-Y|}{1+|X-Y|}\right]$$
Show that
   a) $d(X,Y)$ is a *metric*, *i.e.*, that it satisfies the three rules
      - $d(X,Y) = d(Y,X)$
      - $d(X,Y) \geq 0$, and $d(X,Y) = 0$ if and only if $X = Y$ *a.s.*
      - $d(X,Z) \leq d(X,Y) + d(Y,Z)$
   b) $d(X_n, X) \to 0$ as $n \to \infty$ if and only if $X_n \to X$ in probability

2. Let X and $\{X_n\}$ be random variables on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$. In class we showed that, if $X_n \to X$ in probability then $X_{n_i} \to X$ along some subsequence n_i. In fact something slightly stronger is true: Show that $X_n \to X$ in probability if and only if every subsequence n_i has a further subsequence n_{i_j} such that $X_{n_{i_j}} \to X$ almost surely. Note that the "only if" part follows directly from our in-class result, but the "if" part is new.

3. Using 2. above, give a two-line proof that if $X_n \to X$ in probability, then also $X_n \to X$ in distribution. (Hint: Lebesgue's DCT).

4. If $X_n \to X$ almost surely, does it follow that $\{X_n\}$ is Uniformly Integrable? Give a proof or counterexample.

5. Let U have the uniform distribution on $[0, 1)$ and define a sequence of random variables by
$$X_n = n\, U^{n^2}$$
In which (if any) of the following ways does X_n converge, and to what limit? Why?

   ○ *pr.* ○ *a.s.* ○ L^1 ○ L^2 ○ L^∞ ○ dist