STA244 **2/5/2005**

Homework 3

Due 2/12/2001

1. Recall from class that a non-central $\chi^2(m, \delta)$ can be represented as a Poisson mixture of central χ^2 random variables, where $Y \sim P(\delta/2)$ and $X|Y \sim \chi^2(m + 2y, 0)$. Find the mean and variance of a non-central Chi-squared random variable with m degrees of freedom and non-centrality parameter δ.

2. Consider the linear model $Y = \mu + \epsilon$ where $\mu = X\beta$, X is $n \times p$ rank $r \le p$ and $\epsilon \sim N(0, \sigma^2 I_n)$

   (a) Show that if X is of full column rank $r = p$, then $P_X = X(X'X)^{-1}X'$ is an orthogonal projection onto the space spanned by the columns of X.

   (b) Find an expression for the projection P_X in the non-full rank case (*hint: use the Singular Value Decomposition Thm with* X).

   (c) Show that $Q_X \equiv I - P_X$ is also an orthogonal projection on to the orthogonal complement of the span of X, $S(X)^\perp$.

   (d) Find the distribution of $||P_X Y||^2$. (general rank case)

   (e) Find the distribution of $||Q_X Y||^2$. (general rank case)

   (f) Find the distribution of $||P_X Y||^2/||Q_X Y||^2)$.

3. Suppose we have a $n \times p$ matrix Q and a $p \times p$ upper triangular matrix R such that $Q'Q = I_p$ and $QR = X$. *assume that* X *is of rank* p

   (a) Show that $R'R = X'X$.

   (b) Show that $\hat{\beta} = R^{-1}Q'Y$. Thus to compute $\hat{\beta}$, one computes $z = Q'Y$ then solves the system of equations $R\hat{\beta} = z$ by back substitution without explicit inversion of $X'X$.

   (c) Show that QQ' is an orthogonal projection of rank p onto the $S(X)$ and that $\hat{Y} = QQ'Y$.

   (d) Find e (the residuals) and residual sum of squares in terms of Y and Q.

   (e) Show that the variance of a linear combination $c'\hat{\beta}$ can be written as $\sigma^2 d'd$ where $d = R^{-T}c$, where $R^{-T} = (R')^{-1}$. To find d one can use back substitution in the system of equations $R'd = c$.

   (f) Suppose that

   $$R = \begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & 8 \end{bmatrix}$$

   Assuming that $\sigma^2 = 1$, find the variance of $\hat{\beta}_0$, the variance of $\hat{\beta}_1$ and the covariance between $\hat{\beta}_0$ and $\hat{\beta}_1$ using back substitution. The latter requires a slight extension of the above results.