STA244 **2/28/2005**

Homework 5

Due 3/7/2001

1. For a random vector $\epsilon \in \Re^n$, ϵ is called exchangeable if ϵ has the same distribution as any permutation of the vector ϵ. If ϵ is exchangeable, prove that $E(\epsilon) = \mathbf{1}\delta$ ($\delta \in \Re$), and that the $\mathrm{Cov}(\epsilon) = \Sigma$ has the form

$$\Sigma = \begin{bmatrix} a & & & \\ & a & b & \\ & b & \ddots & \\ & & & a \end{bmatrix}$$

i.e. $\mathrm{var}(\epsilon_i) = a$ for all i and $\mathrm{cov}(\epsilon_i, \epsilon_j) = b$, $i \neq j$. Σ is said to have the form of an intra-class correlation matrix.

2. (a) For $Y \in \Re^n$, assume that $E(Y) = \mu \in M$ and $Y = \mu + \epsilon$, where ϵ has an exchangeable distribution with $\delta = 0$. Show that Σ can be written as $\alpha P_e + \beta Q_e$, $\alpha > 0$, $\beta > 0$, where P_e is the projection onto the space $e = S(1)$, the space spanned by the n-dimensional vector of ones; $Q_e = I_n - P_e$.

   (b) Prove that Σ is nonsingular iff $\alpha \neq 0$ and $\beta \neq 0$. (what does this imply about conditions on a and b?) Show that $\Sigma^{-1} = (1/\alpha)P_e + (1/\beta)Q_e$ (for $\Sigma > 0$).

   (c) If $e \in M$ or $e \in M^{\perp}$, show that the Gauss-Markov and least squares estimators for μ are the same for each α and β.

   (d) If $e \notin M$ and $e \notin M^{\perp}$, show that there are values of α and β so that the least squares and Gauss-Markov estimators of μ differ.

   (e) If $Y \sim N(\mu, \Sigma)$ with Σ having the above form, and $M \subseteq (S(e))^{\perp}$ ($M \neq (S(e))^{\perp}$), find the maximum likelihood estimates for μ, α and β. What happens when $M = S(e)$?