Concepts of Point Estimate

- A **point estimate** of some population parameter θ is a numerical value calculated from the sample.

- A **point estimator** is a formula or rule that tells us how to calculate a numerical estimate, denoted by $\widehat{\theta}(y_1, y_2, \ldots, y_n)$.

- The **bias B** of an estimator $\widehat{\theta}$ is equal to

$$B = \mathbb{E}(\widehat{\theta}) - \theta$$

- An estimator $\widehat{\theta}$ is **unbiased** if $\mathbb{E}(\widehat{\theta}) = \theta$, i.e., $B = 0$.

- The **mean squared error** of a point estimator is equal to

$$\begin{aligned} \mathbb{E}[(\widehat{\theta} - \theta)^2] &= \mathbb{E}[(\widehat{\theta} - \mathbb{E}(\widehat{\theta}) + \mathbb{E}(\widehat{\theta}) - \theta)^2] \\ &= \mathrm{Var}(\widehat{\theta}) + B^2 \end{aligned}$$

Method of Moment

- Let $y_1, y_2, \ldots, y_n$ represent a random sample of size n from some distribution.

  kth population moment: $\mathbb{E}(y^k)$

  kth sample moment:

$$m^k = \frac{y_1^k + y_2^k + \cdots + y_n^k}{n}$$

- Suppose the population distribution has parameters $\theta_1, \ldots, \theta_m$. Then the **moment estimators**, $\widehat{\theta}_1, \ldots, \widehat{\theta}_m$, are obtained by equating the first m sample moments to the corresponding first m population moments and solving the resulting equations for the unknown parameters.

Examples: Find the moment estimators.

- $y_1, y_2, \ldots, y_n \sim$ Exp(β). ($\hat{\beta} = \bar{y}$)
- $y_1, y_2, \ldots, y_n \sim$ Poisson(λ). ($\hat{\lambda} = \bar{y}$)
- $y_1, y_2, \ldots, y_n \sim$ Gamma(α, β). (see example 8.3 on page 346)
- $y_1, y_2, \ldots, y_n \sim N(\mu, \sigma^2)$.

$$\hat{\mu} = \bar{y} \quad \hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n}.$$

Note: The moment estimator of σ^2 is not an unbiased estimator.

Method of Maximum Likelihood

- Suppose we randomly select a sample of n observations, $y_1, \ldots, y_n$ from a distribution $p(y \mid \theta)$, where θ is an unknown parameter. Then the **likelihood** of the sample is

$$L(\theta) = p(y_1 \mid \theta) \cdot p(y_2 \mid \theta) \cdots p(y_n \mid \theta)$$

Note:

  - The likelihood is the joint probability function $p(y_1, \ldots, y_n \mid \theta)$ when y_i's are discrete r.v.
  - The likelihood is the joint density function $f(y_1, \ldots, y_n \mid \theta)$ when y_i's are continuous r.v.
  - Once we have observed y_i's, the likelihood is a function of only the unknown parameter θ.

- **Ronald A. Fisher** (1890-1962):

  One should choose as an estimate of θ the value of θ that maximizes the likelihood $L(\theta)$.

- MLE $\widehat{\theta} = \text{argmax}_\theta \; L(\theta)$

- MLE $\widehat{\theta} = \text{argmax}_\theta \; \log L(\theta)$

$$
\begin{aligned}
L(\theta) &= \prod_{i=1}^{n} p(y_i \mid \theta) \\
\log L(\theta) &= \sum_{i=1}^{n} \log p(y_i \mid \theta)
\end{aligned}
$$

How to Find MLE

- MLE $\hat{\theta} = \text{argmax}_{\theta}\ L(\theta)$.

- Solve
$$\frac{dL}{d\theta} = 0 \qquad \text{or} \quad \frac{d \log L}{d\theta} = 0$$
and check that the resulting solution is a maximum.

- Examples:

  (1) (example 8.4) Find the MLE of Exp(β).

  (2) (example 8.5) Find the MLE of Normal(μ, σ^2).

Complications in Using MLE

- Maximum occurs at a discontinuous point.

  Example: Suppose $y1, \ldots, y_n \sim$ Unif$(0, \theta)$. Find the MLE of θ.

- Close-form solution does not exist

  Example: Suppose $y1, \ldots, y_n \sim$ Gamma(α, β). Find the MLE.

Confidence Interval

- **Aim:** How to use the sample to calculate two numbers that define an interval that will enclose the unknown parameter with certain probability (confidence).

- The resulting random interval is called a **confidence interval**.

- The probability that the interval contains the unknown parameter is called its **confidence coefficient**.

$$P(\text{LCL} \leq \theta \leq \text{UCL}) = 1 - \alpha$$

LCL$(y_1, \ldots, y_n)$: lower confidence limit

UCL$(y_1, \ldots, y_n)$: upper confidence limit

Case 1: Normal with Known Variance

Suppose $\widehat{\mu} \sim N(\mu, \sigma^2)$ with σ^2 known. Define

$$z = \frac{\widehat{\mu} - \mu}{\sigma} \sim N(0, 1).$$

Locate values $z_{\alpha/2}$ and $-z_{\alpha/2}$ that place a probability of $\alpha/2$ in each tail of $N(0, 1)$. For example, $z_{.025} = 1.96$.

$$\begin{aligned}
1 - \alpha &= P(-z_{\alpha/2} \le z \le z_{\alpha/2}) \\
&= P(-z_{\alpha/2} \le \frac{\widehat{\mu} - \mu}{\sigma} \le z_{\alpha/2}) \\
&= P(-z_{\alpha/2}\sigma \le \widehat{\mu} - \mu \le z_{\alpha/2}\sigma) \\
&= P(\widehat{\mu} - z_{\alpha/2}\sigma \le \mu \le \widehat{\mu} + z_{\alpha/2}\sigma)
\end{aligned}$$

Theorem 8.2 Let $\widehat{\mu} \sim N(\mu, \sigma^2)$. Then a $(1 - \alpha)100\%$ confidence interval for μ is

$$\widehat{\mu} - z_{\alpha/2}\sigma \text{ to } \widehat{\mu} + z_{\alpha/2}\sigma$$

Example : ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch(CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at 60^0C are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2 and 64.3. Assume that impact energy is normally distributed with $\sigma = 1J$.

Find a 95% CI for μ, the mean impact energy.

$$\hat{\mu} = \bar{y} \sim N(\mu, \sigma^2/n)$$

$$n = 10, \quad \sigma = 1, \quad \alpha_{\alpha/2} = z_{2.5\%} = 1.96$$

$$\begin{aligned} \bar{y} - z_{\alpha/2}\sigma_{\bar{y}} \leq & \quad \mu \quad \leq \bar{y} + z_{\alpha/2}\sigma_{\bar{y}} \\ 64.46 - 1.96\frac{1}{\sqrt{10}} \leq & \quad \mu \quad \leq 64.46 + 1.96\frac{1}{\sqrt{10}} \\ 63.84 \leq & \quad \mu \quad \leq 65.08 \end{aligned}$$

How many specimens must be tested to ensure that the 95% CI of μ has a length of at most $1.0J$?

$$n = [\frac{(1.96)1}{0.5}]^2 = 15.37.$$

Interpreting a CI

- Can we conclude: The true mean μ is within the interval (63.84, 65.08) with probability 0.95? – **NO**

- The statement $63.84 \leq \mu \leq 65.08$ is either correct (true with probability 1) or incorrect (false with probability 1).

- Remember that a CI is a **random interval** and the correct interpretation of a $100(1-\alpha)\%$ CI should depend on the relative frequency view of probability.

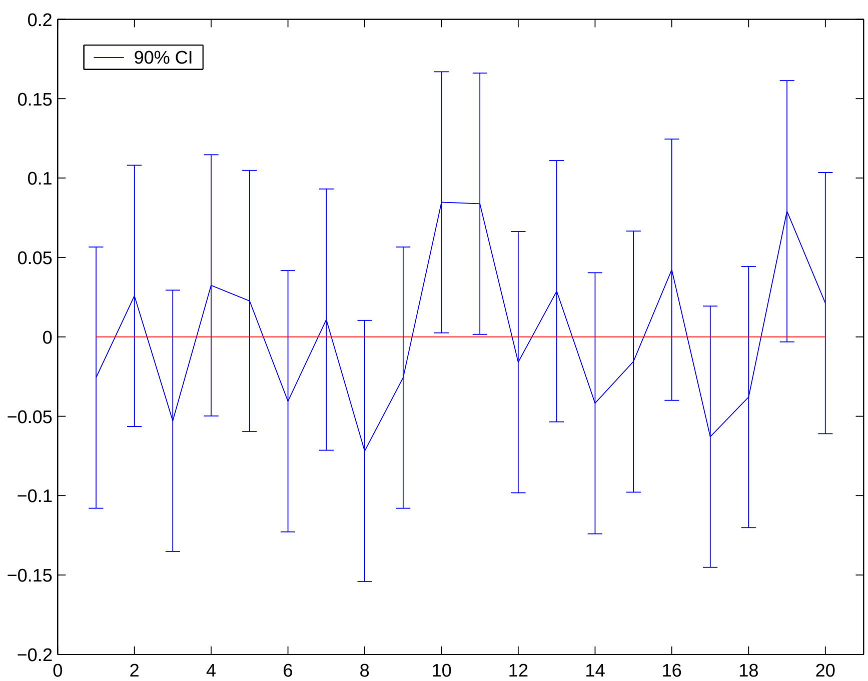
- We conclude:

  If we were to repeatedly collect a sample of size n and construct a 95% CI for each sample, then we expect 95% of the intervals to enclose the true parameter μ.

```
m=20; n=10;
y = normrnd(0, 0.5, m,n);
y_mean = mean(y,2);
e = norminv(0.95)*0.5/sqrt(n)*ones(m,1)
errorbar(1:m, y_mean, e);
h = line([1, m], [0, 0]);
set(h, 'color', [1 0 0]);

LCL = y_mean - e;
UCL = y_mean + e;
sum(LCL > 0) + sum(UCL < 0)

ans =
     2
```



- If $\bar{y}$ is the sample mean of a random sample of size n from $N(\mu, \sigma^2)$, the $(1-\alpha)100\%$ CI is

$$\bar{y} - z_{\alpha/2}\sigma/\sqrt{n} \le \mu \le \bar{y} + z_{\alpha/2}\sigma/\sqrt{n}$$

- The length of the CI is equal to $2z_{\alpha/2}\sigma/\sqrt{n}$.

- What's the relationship between the length of a CI and
  - the confidence coefficient $(1-\alpha)100\%$?
  - the sample size n?

    Q: How many sample size we should choose in order to have the length of CI less than l_0.

$$2z_{\alpha/2}\sigma/\sqrt{n} \le l_0$$

$$n \ge \left(\frac{z_{\alpha/2}\sigma}{l_0/2}\right)^2$$

Sampling Dist Related to Normal

A random sample $y_1, y_2, \ldots, y_n$ is drawn from $N(\mu, \sigma^2)$.

$$\text{sample mean } \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\text{sample var } s^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}$$

- Recall that $\chi^2(\nu) = z_1^2 + z_2^2 + \cdots + z_\nu^2$, where each $z_i \sim N(0, 1)$.

- $(n-1)s^2/\sigma^2 \sim \chi^2(n-1)$

- Let z be a standard normal and χ^2 be a chi-square with ν degrees of freedom. If z and χ^2 are independent, then

  $$t = \frac{z}{\sqrt{\chi^2/\nu}}$$

  has a **Student's t distribution** with ν degree of freedom.

Case 2: Normal with Unknown Variance (Example 8.6) Let $\bar{y}$ and s^2 be the sample mean and variance based on a random sample of n normal(μ, σ^2) observations

Define

$$
\begin{aligned}
t &= \frac{\bar{y}-\mu}{s/\sqrt{n}} = \frac{\frac{\bar{y}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)s^2}{\sigma^2}}/\sqrt{n-1}} \\
&\sim \text{ Student's t distribution}
\end{aligned}
$$

$$
\begin{aligned}
& 1-\alpha \\
= & P(-t_{\alpha/2,n-1} \leq t \leq t_{\alpha/2,n-1}) \\
= & P(-t_{\alpha/2,n-1} \leq \frac{\bar{y}-\mu}{s/\sqrt{n}} \leq t_{\alpha/2,n-1}) \\
= & P(\bar{y} - t_{\alpha/2,n-1}\Big(\frac{s}{\sqrt{n}}\Big) \leq \mu \leq \bar{y} + t_{\alpha/2,n-1}\Big(\frac{s}{\sqrt{n}}\Big))
\end{aligned}
$$

Take a look of Example 8.7