Complications in Using MLE

- Maximum occurs at a discontinuous point.

  Example: Suppose $y_1, \ldots, y_n \sim \text{Unif}(0, \theta)$. Find the MLE of θ.

$$\begin{aligned} L(\theta) &= \prod_{i=1}^{n} \frac{1}{\theta} \mathbf{1}_{\{0 \le y_i \le \theta\}} \\ &= \frac{1}{\theta^n} \mathbf{1}_{\{\min(y_i) \ge 0\}} \mathbf{1}_{\{\max(y_i) \le \theta\}} \\ &= \begin{cases} 0 & \theta < \max(y_i) \\ \frac{1}{\theta^n} & \theta \ge \max(y_i) \end{cases} \end{aligned}$$

  So $\hat{\theta} = \text{argmax}_\theta L(\theta) = \max(y_i)$.

  $\hat{\theta} = \max y_i$ **is biased. Why?**

- Close-form solution does not exist

  Example: Suppose $y_1, \ldots, y_n \sim \text{Gamma}(\alpha, \beta)$. Find the MLE.

Confidence Interval

- **Aim:** How to use the sample to calculate two numbers that define an interval that will enclose the unknown parameter with certain probability (confidence).

- The resulting random interval is called a **confidence interval**.

- The probability that the interval contains the unknown parameter is called its **confidence coefficient**.

$$P(\text{LCL} \le \theta \le \text{UCL}) = 1 - \alpha$$

  α is usually small, for example, 5% or 2.5%.

  LCL$(y_1, \ldots, y_n)$: lower confidence limit

  UCL$(y_1, \ldots, y_n)$: upper confidence limit

Case 1: Normal with Known Variance

Suppose $\widehat{\mu} \sim N(\mu, \sigma^2)$ with σ^2 known.

(1) Define (pivotal statistic)

$$z = \frac{\widehat{\mu} - \mu}{\sigma} \sim N(0, 1).$$

(2) Locate values $z_{\alpha/2}$ and $-z_{\alpha/2}$ that place a probability of $\alpha/2$ in each tail of $N(0,1)$. For example, $z_{.025} = 1.96$.

$$\begin{aligned} 1 - \alpha &= P(-z_{\alpha/2} \leq z \leq z_{\alpha/2}) \\ &= P(-z_{\alpha/2} \leq \frac{\widehat{\mu} - \mu}{\sigma} \leq z_{\alpha/2}) \\ &= P(-z_{\alpha/2}\sigma \leq \widehat{\mu} - \mu \leq z_{\alpha/2}\sigma) \\ &= P(\widehat{\mu} - z_{\alpha/2}\sigma \leq \mu \leq \widehat{\mu} + z_{\alpha/2}\sigma) \end{aligned}$$

Theorem 8.2 Let $\widehat{\mu} \sim N(\mu, \sigma^2)$. Then a $(1 - \alpha)100\%$ confidence interval for μ is

$$\widehat{\mu} - z_{\alpha/2}\sigma \text{ to } \widehat{\mu} + z_{\alpha/2}\sigma$$

Example : ASTM Standard E23 defines standard test methods for notched bar impact testing of metallic materials. The Charpy V-notch(CVN) technique measures impact energy and is often used to determine whether or not a material experiences a ductile-to-brittle transition with decreasing temperature. Ten measurements of impact energy (J) on specimens of A238 steel cut at 60^0C are as follows: 64.1, 64.7, 64.5, 64.6, 64.5, 64.3, 64.6, 64.8, 64.2 and 64.3. Assume that impact energy is normally distributed with $\sigma = 1J$.

Find a 95% CI for μ, the mean impact energy.

$$\widehat{\mu} = \bar{y} \sim N(\mu, \sigma^2/n)$$

$$n = 10, \quad \sigma = 1, \quad \alpha_{\alpha/2} = z_{2.5\%} = 1.96$$

$$\begin{aligned} \bar{y} - z_{\alpha/2}\sigma_{\bar{y}} \leq & \quad \mu \quad \leq \bar{y} + z_{\alpha/2}\sigma_{\bar{y}} \\ 64.46 - 1.96\frac{1}{\sqrt{10}} \leq & \quad \mu \quad \leq 64.46 + 1.96\frac{1}{\sqrt{10}} \\ 63.84 \leq & \quad \mu \quad \leq 65.08 \end{aligned}$$

How many specimens must be tested to ensure that the 95% CI of μ has a length of at most $1.0J$?

$$n = [\frac{(1.96)1}{0.5}]^2 = 15.37.$$

Interpreting a CI

- Can we conclude: The true mean μ is within the interval (63.84, 65.08) with probability 0.95? $-$ **NO**

- The statement $63.84 \leq \mu \leq 65.08$ is either correct (true with probability 1) or incorrect (false with probability 1).

- Remember that a CI is a **random interval** and the correct interpretation of a $100(1-\alpha)\%$ CI should depend on the relative frequency view of probability.

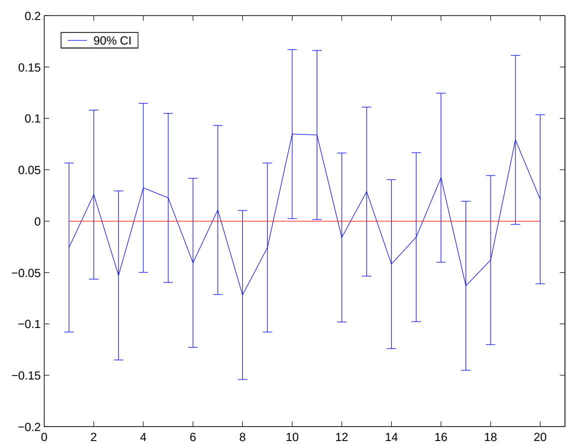
- We conclude:

  If we were to repeatedly collect a sample of size n and construct a 95% CI for each sample, then we expect 95% of the intervals to enclose the true parameter μ.


```
m=20; n=10;
y = normrnd(0, 0.5, m,n);
y_mean = mean(y,2);
e = norminv(0.95)*0.5/sqrt(n)*ones(m,1)
errorbar(1:m, y_mean, e);
h = line([1, m], [0, 0]);
set(h, 'color', [1 0 0]);

LCL = y_mean - e;
UCL = y_mean + e;
sum(LCL > 0) + sum(UCL < 0)

ans =
     2
```



- If $\bar{y}$ is the sample mean of a random sample of size n from $N(\mu, \sigma^2)$, the $(1-\alpha)100\%$ CI is

$$\bar{y} - z_{\alpha/2}\sigma/\sqrt{n} \leq \mu \leq \bar{y} + z_{\alpha/2}\sigma/\sqrt{n}$$

- The length of the CI is equal to $2z_{\alpha/2}\sigma/\sqrt{n}$.

- What's the relationship between the length of a CI and
  - the confidence coefficient $(1-\alpha)100\%$?
  - the sample size n?

    Q: How many sample size we should choose in order to have the length of CI less than l_0.

$$2z_{\alpha/2}\sigma/\sqrt{n} \leq l_0$$

$$n \geq \left(\frac{z_{\alpha/2}\sigma}{l_0/2}\right)^2$$

Sampling Dist Related to Normal

A random sample $y_1, y_2, \ldots, y_n$ is drawn from $N(\mu, \sigma^2)$.

$$\text{sample mean } \bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$$

$$\text{sample var } s^2 = \frac{\sum_{i=1}^{n}(y_i - \bar{y})^2}{n-1}$$

- Recall that $\chi^2(\nu) = z_1^2 + z_2^2 + \cdots + z_\nu^2$, where each $z_i \sim N(0,1)$.

- $(n-1)s^2/\sigma^2 \sim \chi^2(n-1)$

- Let z be a standard normal and χ^2 be a chi-square with ν degrees of freedom. If z and χ^2 are independent, then

$$t = \frac{z}{\sqrt{\chi^2/\nu}}$$

  has a **Student's t distribution** with ν degree of freedom.

Case 2: Normal with Unknown Variance

(Example 8.6) Let $\bar{y}$ and s^2 be the sample mean and variance based on a random sample of n normal(μ, σ^2) observations

Start with the pivotal statistic

$$\begin{aligned} t &= \frac{\bar{y}-\mu}{s/\sqrt{n}} = \frac{\frac{\bar{y}-\mu}{\sigma/\sqrt{n}}}{\sqrt{\frac{(n-1)s^2}{\sigma^2}/(n-1)}} \\ &\sim \text{Student's t distribution} \end{aligned}$$

$$\begin{aligned} & 1-\alpha \\ = \; & P(-t_{\alpha/2,n-1} \leq t \leq t_{\alpha/2,n-1}) \\ = \; & P(-t_{\alpha/2,n-1} \leq \frac{\bar{y}-\mu}{s/\sqrt{n}} \leq t_{\alpha/2,n-1}) \\ = \; & P(\bar{y} - t_{\alpha/2,n-1}\Big(\frac{s}{\sqrt{n}}\Big) \leq \mu \leq \bar{y} + t_{\alpha/2,n-1}\Big(\frac{s}{\sqrt{n}}\Big)) \end{aligned}$$

Take a look of Example 8.7