Probability theory is a systematic method to describe uncertainty and randomness

Probability is generally used synonymously with "chance", "odds" and other similar concepts.

Feng said: *"If a fair coin is tossed, the probability of observing a head is 50% or 1/2"*.

What do we mean when we say that the probability of a head is 1/2? – In a very long series of tosses, approximately half would result in a head.

An **experiment** is the process of obtaining an observation or taking a measurement.

A **simple event** is a basic outcome of an experiment; it can not be decomposed into simpler outcomes.

The **sample space** of an experiment is the collection of all its simple events.

An **event** is a specific collection of simple events.

Two special events: the empty set $\emptyset$ and the complete sample space S.

Example: Consider the experiment of tossing two coins.

- What is the sample space?

- Draw the event A $=\{$observe at least one head$\}$ in **Venn diagram**

- Calculate $P(A)$.

The **probability** of a simple event is a number that measures the likelihood that the event will occur when the experiment is performed.

The probability of a simple event E is denoted by $P(E)$.

If an experiment has n different outcomes and they are all equally likely, then the probability of each outcome is equal to $1/n$.

The probability $P(E)$ can be *approximated* by the proportion of times that E is observed when the experiment is repeated a very large number of times.

Rules for Assigning Probabilities

Let $E_1, E_2, \ldots, E_k$ be all the simple events in a sample space.

1. All probabilities *must* lie between 0 and 1:

$$0 \leq P(E_i) \leq 1.$$

Specially $P(\emptyset) = 0$ and $P(S) = 1$.

2. If an event $A = \{E_1, E_2\}$ then

$$P(A) = P(E_1) + P(E_2)$$

3. The sum of the probabilities of all the simple events within a sample space must be equal to 1.

$$\sum_{i=1}^{n} P(E_i) = 1.$$

Calculate Probabilities of Events

0. Define the experiment, i.e., describe the process used to make an observation and the type of observation that will be recorded.

1. List the simple events, i.e., find the sample space.

2. Assign probabilities to the simple events.

3. Determine the collection of simple events contained in the event of interest.

4. Sum the simple event probabilities to get the event probability.

Example 3.4 on Page 87: A computer programmer must select three jobs from among five jobs awaiting the programmer's attention. If, unknown to the programmer, the jobs vary in the length of programming time required, what is the probability that:

a. The programmer selects the two jobs that require the least amount of time?

b. The programmer selects the job that require the most time?

Compound Events

- **Intersection**

$$A \cap B = \{\text{A occurs and B occurs}\}$$

- **Union**

$$A \cup B = \{\text{A occurs or B occurs}\}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

- **Complement**

$$A^c = \{\text{A does not occur}\}$$

$P(A) + P(A^c) = 1$, i.e., $P(A^c) = 1 - P(A)$.
Note: $A \cap A^c = \emptyset$; $\quad A \cup A^c = S$.

Example 3.5 Consider the die-tossing experiment. Define the following events:

A: Toss an even number

B: Toss a number less than or equal to 3

What are the probabilities for events $A \cup B$, $A \cap B$, A^c and $(A \cup B) \cap (A^c \cap B^c)$?

Distribution Law for Events

- In high-school algebra, we learn the distribution law:

$$(A + B) \times C = A \times C + B \times C.$$

- For sets (hence, for events) we can replace $+$ with $\cap$ and $\times$ with $\cup$, or vice versa, leading to the two distribution law for events:

$$\begin{aligned} (E_1 \cap E_2) \cup E_3 &= (E_1 \cup E_3) \cap (E_2 \cup E_3) \\ (E_1 \cup E_2) \cap E_3 &= (E_1 \cap E_3) \cup (E_2 \cap E_3) \end{aligned}$$

DeMorgan Law

- First Law:

$$(E_1 \cup E_2)^c = E_1^c \cap E_2^c$$

  "Neither E_1 nor E_2 occurs" is equivalent to "E_1 does not occur and E_2 does not occur".

- Second Law:

$$(E_1 \cap E_2)^c = E_1^c \cup E_2^c$$

  "E_1 and E_2 do not occur together" is equivalent to "E_1 does not occur or E_2 does not occur".

Example 3.5 (Cont.) Consider the die-tossing experiment. Define the following events:

A: Toss an even number

B: Toss a number less than or equal to 3

Given that you've tossed a number less than or equal to 3, what is the probability of tossing an even number?

What's the probability of A occurs given that B occurs?

Conditional Probability

The **conditional probability** that event A occurs given that event B occurs is defined to be

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)},$$

where $P(B) > 0$.

The Prisoner's Dilemma: Three prisoners, A, B and C, with apparently equally good records have applied for parole. The parole board has decided to release two of them, and the prisoners know this but not which two. A warder friend of prisoner A knows who are going to be released. Prisoner A realizes that he could not ask his friend whether he, A, is to be released, but he could ask for the name of one prisoner *other than himself* who is to be released.

Prisoner A thinks: Before he asks, his chance of release is 2/3. If the warder says "B will be released", then his chance will go down to 1/2. So A decides not to reduce his chance by asking.

Anything wrong here?

Show that prisoner A's calculation for conditional probability is wrong. Given what the warder says, A's chance of being released still remains 2/3.