Probability Rules for $\cap$ and $\cup$

- **Additive Rule**

  $$P(A \cup B) = P(A) - P(B) - P(A \cap B).$$

  A and B are **mutually exclusive** if

  $$A \cap B = \emptyset$$

  $$\Longrightarrow P(A \cup B) = P(A) + P(B).$$

- **Multiplicative Rule**

  $$P(A \cap B) = P(A \,|\, B)P(B) = P(B \,|\, A)P(A).$$

  A and B are **independent** if

  $$P(A \,|\, B) = P(A),$$

  $$\Longrightarrow P(A \cap B) = P(A)P(B).$$

  $$\Longrightarrow P(B \,|\, A) = P(B).$$

  Events are not independent are said to be **dependent**.

Ex 3.35 on page 118: The computing system at a large university is currently undergoing shutdown for repairs. Previous shutdowns have been due to hardware failure, software failure or power failure (those failures do not happen together). The system is forced to shut down 73% of the time when it experiences hardware problems, 12% of the time when it experiences software problems, and 88% of the time when it experiences electronic problems. Maintenance engineers have determined that the probabilities of hardware, software and power problems are 0.01, 0.05, and 0.02.

What is the probability that the current shutdown is due to the hardware failure?

$$\begin{aligned} P(\text{Hardware}) &= 0.01 & \quad P(\text{Shutdown} \,|\, H) &= 0.73 \\ P(\text{Software})) &= 0.05 & \quad P(\text{Shutdown} \,|\, S) &= 0.12 \\ P(\text{Power}) &= 0.02 & \quad P(\text{Shutdown} \,|\, P) &= 0.88 \end{aligned}$$

If Shutdown occurs, one of H, S, P occurs. And $H \cap S \cap P = \emptyset$.

$$P(\text{Hardware} \,|\, \text{Shutdown}) = ?$$

$$P(\text{Hardware} \,|\, \text{Shutdown}) = \frac{P(\text{ Hardware and Shutdown})}{P(\text{Shutdown})}$$

$$\begin{aligned} P(\text{ Hardware} \cap \text{Shutdown}) &= P(\text{Shutdown} \,|\, H)P(\text{Hardware}) \\ &= 0.73 \times 0.01 \end{aligned}$$

$$\begin{aligned} P(\text{Shutdown}) &= P(\text{Shutdown, either H, S or P}) \\ &= P(\text{Shutdown} \cap (H \cup S \cup P)) \\ &= P(\text{Shutdown} \cap H) + P(\text{Shutdown} \cap S) \\ &\quad + P(\text{Shutdown} \cap P) \\ &= 0.73 \times 0.01 + 0.12 \times 0.05 + 0.88 \times 0.02 \\ &= 0.0309 \end{aligned}$$

Bayes' Rule

Given k mutually exclusive events, $A_1, A_2, \dots A_k$ and an observed event E. If

$$E = E \cap (A_1 \cup A_2 \cup \cdots \cup A_k),$$

we have

$$\begin{aligned} P(A_i \,|\, E) &= \frac{P(A_i \cap E)}{P(E)} \\ &= \frac{P(E \,|\, A_i)P(A_i)}{P(E \,|\, A_1)P(A_1) + \cdots + P(E \,|\, A_k)P(A_k)}, \end{aligned}$$

where we use

$$\begin{aligned} P(A_i \cap E) &= P(E \,|\, A_i)P(A_i) \\ P(E) &= P(E \cap A_1) + \cdots + P(E \cap A_k) \\ &= P(E \,|\, A_1)P(A_1) + \cdots + P(E \,|\, A_k)P(A_k) \end{aligned}$$

- Let $A_1, A_2, \ldots, A_k$ be k **mutually exclusive and exhaustive events**, that is, these k events *partition* the entire sample space into non-overlapping parts,

$$A_i \cap A_j = \emptyset, \quad i \neq j$$
$$A_1 \cup A_2 \cdots \cup A_k = S$$

- For example, $k = 2$, $A_1 = E$, $A_2 = E^c$.

- For any event E, we have

$$\begin{aligned} P(E) &= P(E \cap (A_1 \cup A_2 \cup \ldots A_k)) \\ &= \sum_{i=1}^{k} P(E \cap A_i) \\ &= \sum_{i=1}^{k} P(E \mid A_i) P(A_i). \end{aligned}$$

Ex 3.35 (cont.) Which (hardware, software and power) failure is most likely responsible for the current shutdown?

Notice that

$$\begin{aligned} P(\star \mid \text{Shutdown}) &= \frac{P(\star \cap \text{Shutdown})}{P(\text{Shutdown})} \\ &\sim P(\star \cap \text{ Shutdown}) \\ &= P(\text{Shutdown} \mid \star) P(\star), \end{aligned}$$

where $\star$ denotes for Hardware, software or power.

So to do the comparison, we just need to calculate the numerators.

$$\begin{aligned} \text{Hardware} &: 0.73 \times 0.01 = 0.0073 \\ \text{Software} &: 0.12 \times 0.05 = 0.0060 \\ \text{Power} &: 0.88 \times 0.02 = 0.0176 \end{aligned}$$

Some Examples

Example 3.22 Consider an experiment that consists of tossing a coin 10 times. How many different simple events for this experiment?

Example 3.23 Suppose there are five different space flights scheduled, each requiring one astronaut. Assuming that no astronaut can go on more than one space flight, in how many different ways can five of the country's top 100 astronauts be assigned to the five space flights?

Example 3.4 In how many different ways can a computer programmer select 3 jobs from among 5 jobs who vary in the length of programming time required?

Example 3.26 You have 12 system analysts and you want to assign three to job 1, four to job 2 and five to job 3. In how many different ways can you make this assignment?

The Multiplicative Rule

You have k sets of elements, n_1 in the first set, n_2 in the second set, ... , and n_k in the kth set. Suppose you want to form a sample of k elements by taking one element from each of the k sets. The total number of different samples that can be formed is the product

$$n_1 n_2 \cdots , n_k.$$

Example: If I can get either a chicken or tuna sandwich, with either Coke or ice tea, and either fries, chips or onion rings, how many possibilities for lunch do I have?

$$\begin{aligned} (2 \text{ sandwich choices}) &\times \\ (2 \text{ beverage choices}) &\times \\ (3 \text{ snack choices}) &= 12 \text{ possible meals} \end{aligned}$$

- How many ways I can arrange n **distinct** things?

- n possibilities for the first one, $n-1$ for the second, etc.

- The answer is

$$n(n-1)(n-2)\cdots 2 \times 1 = n!,$$

where $n!$ is called n *factorial*. The quantity 0! is defined to be equal to 1.

Permutations Rule

Given N distinct items and you wish to select n from the N and arrange them within n positions. The total number of all possible choices is equal to

$$P_n^N = N(N-1)(N-2)\cdots(N-n+1) = \frac{N!}{(N-n)!}.$$

Combinations Rule

A sample of n elements is to be chosen from a set of N samples. The number of different samples of n elements is equal to

$$\binom{N}{n} = \frac{N!}{n!(N-n)!}.$$

It is also called *binomial coefficient* because it appears in the binomial theorem of elementary algebra.

Partitions Rule

N distinct elements and you want to partition them into k sets, the first set containing n_1 elements, the second containing n_2 elements, ..., and the kth set containing n_k elements . The number of different partitions is

$$\frac{N!}{n_1! n_2! \cdots n_k!}, \quad n_1 + n_2 + \cdots n_k = N.$$

Example 3.29 A computer rating service is commissioned to rank the top three brands of RGA monitors. A total of 10 brands are to be included in the study.

If the rating service can distinguish no difference among those brands and therefore arrives at the final ranking by chance, what is the probability that company Z's brand is ranked first? In the top three?

Example 3.30 Suppose the computer rating service is to choose the top three EGA monitors from the group of 10, but is *not to rank the three*.

Assuming that the rating service makes its choice by chance and that company X has two brands in the group of ten, what is the probability that exactly one of the company X brands is selected in the top three? At least one?

Random Variables

Consider the experiment of tossing a coin 10 times.

Variable y = the number of heads is observed. The value of y varies in a random manner from one repetition of the experiment to another. $y = 0, 1, \ldots 10$.

To each simple event in the sample space S, there corresponds to one and only one value of the variable y.

Since each possible value of y defines an event, y is called a random variable.

- A **random variable** is a numerical valued function defined over a sample space.

- A **discrete random variable** is one that can assume only a countable number of values (perhaps a finite number).

- The **probability distribution** for a discrete random variable y is a table, graph, or formula that gives the probability $p(y)$ associated with each possible value of y.

  Requirements for a Discrete Probability Distribution

$$0 \leq p(y) \leq 1; \qquad \sum_{\text{all y}} p(y) = 1.$$

Example 4.1 A balanced coin is tossed twice, and the number of y of heads is observed. Find the probability distribution for y.

Let y be a discrete random variable with probability distribution $p(y)$.

- The **mean** or **expected value** of y is

$$\mu = \mathbb{E}(y) = \sum_{\text{all y}} yp(y).$$

- Useful Expectation Theorems

$$\begin{aligned}
(1) &\quad \mathbb{E}(c) = c \\
(2) &\quad \mathbb{E}(cy) = c\mathbb{E}(y) \\
(3) &\quad \mathbb{E}[g_1(y) + \cdots + g_k(y)] \\
= &\quad \mathbb{E}[g_1(y)] + \cdots + \mathbb{E}[g_k(y)].
\end{aligned}$$

- Let $g(y)$ be a function of y, then the **mean** or **expected value** of $g(y)$ is

$$\mathbb{E}[g(y)] = \sum_{\text{all y}} g(y)p(y).$$

- The **variance** of y is

$$\sigma^2 = \mathbb{E}(y-\mu)^2 = \mathbb{E}(y^2) - \mu^2.$$

- The **standard deviation** of y is

$$\sigma = \sqrt{\sigma^2}.$$