STA244 **1/29/2003**

Homework 3

Due 2/5/2001

1. Recall from class that a non-central $\chi^2(m, \mu^2/2)$ can be represented as a Poisson mixture of independent central χ^2 random variables, where $N \sim P(\mu/2)$ and $X|N \sim \chi^2(m+2N, 0)$. Show that a $\chi^2(m, \mu^2/2)$ is stochastically larger than a central $\chi^2(m, 0)$, i.e. show

$$P(\chi^2(m, 0) > x) \leq P(\chi^2(m, \mu^2/2) > x)$$

2. A (single) non-central $F(m, n, \mu^2/2)$ is equal in distribution to

$$F \sim \frac{\chi^2(m, \mu^2/2)/m}{\chi^2(n, 0)/n}$$

where the numerator and denominator are independent independent χ^2 random variables. Use the Poisson representation to find the mean and variance of a (single) non-central F random variable with m and n degrees of freedom and non-centrality parameter $\mu^2/2$; assume $n > 4$. Recall: *Casella and Berger P. 624*: the mean and variance of a central $F(m, n)$ are $n/(n-2)$ and $2(n/(n-2))^2(m+n-2)/(m(n-4))$ respectively.

3. Consider the linear model $Y = X\beta + \epsilon$ where X is $n \times p$ matrix and $\epsilon \sim N(0, \sigma^2 I_n)$

   (a) Show that if X is of full rank $r(X) = p$, then $P_X = X(X'X)^{-1}X'$ is a rank p orthogonal projection onto the space spanned by the columns of X.

   (b) Show that $Q_X \equiv I - P_X$ is also an orthogonal projection on to the *orthogonal complement* of the span of X, $S(X)^{\perp}$ i.e. if $z \in S(X)^{\perp}$, $z \perp S(X)$. What is the rank of Q_X?.

   (c) Find the distribution of $||P_X Y||^2/||Q_X Y||^2)$.

4. Show that the vectors $Z_1, \ldots X_p$ created by the Gram-Schmidt procedure form an orthonormal basis for the $S(X)$, where the rank of X is p.

$$Y_1 = X_1/(X_1'X_1)^{1/2} \quad (1)$$

$$W_j = W_j - \sum_{k=1}^{j-1}(X_j'Y_k)Y_k \quad (2)$$

$$Y_j = W_j/(W_j'W_j)^{1/2} \quad (3)$$