Final Examination

STA 205: Probability and Measure Theory

Due Tuesday, 2003 Apr 29, 7:00 pm
(or any time before that)

This is an open-book 24-hour take-home examination. You must do your own work— collaboration is not permitted. If a question seems ambiguous or confusing *please* ask me— don't guess, and don't discuss exam questions with others (whether or not they are taking this exam). You can reach me by telephone (w: 684-3275; h: 688-0435) or, better, by e-mail (`wolpert@stat.duke.edu`).

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

This exam is due by 7pm Tuesday, 2003 April 29. You can slip it under my office door (211c Old Chem) or hand it to me earlier.

Print Name: ______________________

Begin Time: __________, April ______, 2003

Finish Time: __________, April ______, 2003

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
6.	/20
Total:	/120

Problem 1: Let $\xi_n \sim \mathsf{Ex}(1)$ be independent random variables with the standard (mean-one) exponential distribution

$$\mathsf{P}[\xi_n \in A] = \int_{A\cap\mathbb{R}_+} e^{-x}\, dx$$

and set

$$X_n = 1_{[n,\infty)}(\xi_n) = \begin{cases} 0 & \xi_n < n \\ 1 & \xi_n \geq n. \end{cases}$$

a. (5) Find the mean $\mu_n = \mathsf{E}[X_n]$ and variance $\sigma_n^2 = \mathsf{Var}[X_n]$ of X_n (*not* of ξ_n):

$\mu_n =$ ______________ $\sigma_n^2 =$ ______________

b. (5) Set $S_n = X_1 + ... + X_n$. Does S_n converge as $n \to \infty$? In what sense? Why?

c. (5) Find the characteristic function of ξ_n:

$\phi(\omega) = \mathsf{E}\left[e^{i\omega\xi_n}\right] =$ ______________

d. (5) Find (an expression for) the characteristic function of S_n:

$\phi_n(\omega) = \mathsf{E}\left[e^{i\omega S_n}\right] =$ ______________

Problem 2: Let $(\Omega, \mathcal{F}, \mathsf{P}) = \big((0,1], \mathcal{B}_1, d\omega\big)$ be the unit interval with Lebesgue measure (length). Define three random variables by

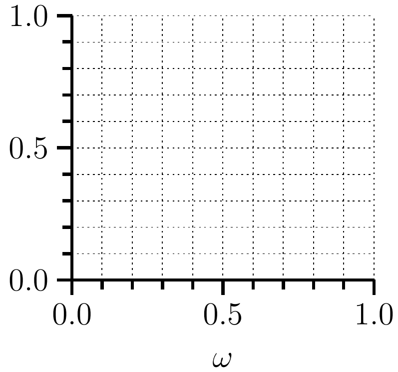
$$X(\omega) = 1_{(0,.6]}(\omega) \qquad Y(\omega) = 1_{(.3,.5]}(\omega) \qquad Z(\omega) = \omega.$$

a. (10) What is the σ-algebra $\sigma(X, Y)$ generated by X and Y?

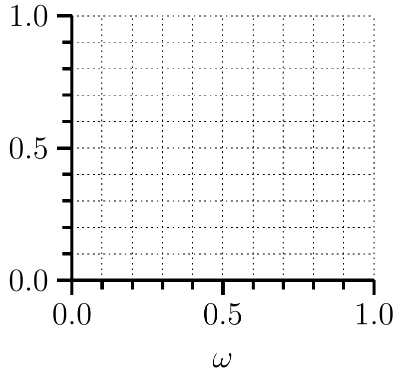
How many events does it include? ____ List them:

$$\sigma(X,Y) = \Bigg\{ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \Bigg\}$$

b. (5) Find and plot the conditional expectation $\mathsf{E}[Z|X, Y]$:



c. (5) Find and plot the conditional expectation $\mathsf{E}[W|X, Y, Z]$ for $W(\omega) \equiv \sqrt{\omega}$ (Hint: No integration will be necessary):

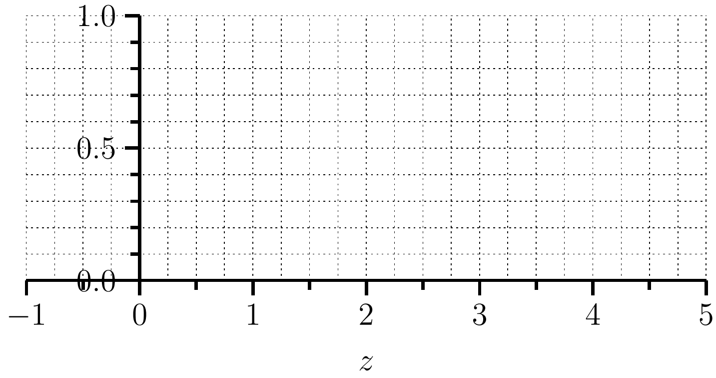


Problem 3: The random variables X and Y have a distribution generated by the following mechanism. A fair coin is tossed; if it falls Heads, then $X = Y = 0$; if it falls Tails, then X and Y are drawn independently from the exponential distribution with mean 1.

a. (5) Are X and Y independent? Prove it.

b. (5) Find and plot the c.d.f. for the sum $Z = X + Y$:

$\mathsf{P}[Z \leq z] =$ ____________________



c. (10) Let $\mathcal{G}$ be the BF generated by Z. Find the conditional expectation of X, given $\mathcal{G} = \sigma(Z)$:

$\mathsf{E}[X|\mathcal{G}] =$ ____________________

Problem 4: (20) Let $\xi_i \sim \mathsf{No}(0,1)$ be independent standard normal random variables and let $S_n \equiv \sum_{i=1}^n \xi_i$ be their partial sum. For real numbers α, β set

$$X_n \equiv e^{\alpha S_n - \beta n}.$$

a. (10) For what pairs $(\alpha, \beta) \in \mathbb{R}^2$ is X_n a *martingale*?
Show your work.

b. (5) For what pairs $(\alpha, \beta) \in \mathbb{R}^2$ are the random variables $\{X_n\}_{n\in\mathbb{N}}$ *uniformly integrable* (not necessarily a martingale)? Why?

c. (5) For $\lambda > 0$ and $\alpha = \beta = 2$, find the best bound you can for:

$\mathsf{P}[\sup_{1\le m\le n} X_m \ge \lambda] \le$ ____________________

What does this say about the Gaussian random walk S_n?

Problem 5: The random variables $\{X_i\}$ are all independent and all satisfy $\mathsf{E}[X_i{}^4] \le 1.0$, but they may have different distributions. Let $S_n \equiv \sum_{i=1}^{n} X_i$ be their partial sum.

a. (10) Does it follow without any further assumptions that S_n/n converges almost surely? Give a proof or counter-example.

b. (10) If in addition we know $\mathsf{E}[X_i] = 0$ and $\mathsf{E}[X_i{}^2] = \sigma^2$ for all i, does it follow without further assumptions that

$$\frac{1}{n}\sum_{m=1}^{n} \sin\left(\frac{S_m}{\sqrt{m}}\right)$$

converges? If so, to what limit and why? No proof is needed, just the basic idea. (+5 Extra credit: Same question with $\cos(\frac{S_m}{\sqrt{m}})$)

Problem 6: A sequence of integrable functions $f_n \in L_1(\mathcal{U}, \mathcal{B}, dx)$ on the unit interval $\mathcal{U} = (0, 1]$ satisfies

$$\lim_{n \to \infty} f_n(x) = f(x)$$

at every point $x \in \mathcal{U}$, for some limit function $f \in L_1(\mathcal{U}, \mathcal{B}, dx)$.

a. (10) Does it follow without any further assumptions that $\int_{\mathcal{U}} f_n(x)\, dx$ converges to $\int_{\mathcal{U}} f(x)\, dx$? Give a proof or a counter-example.

b. (10) If in addition we know that $f_n(x) \ge 0$ and $\int_{\mathcal{U}} f_n(x)\, dx = \int_{\mathcal{U}} f(x)\, dx = 1$, so that $f(x)$ and each $f_n(x)$ is a probability density function, and if we still have $f_n(x) \to f(x)$ for all x, does it follow that random variables $X_n \sim f_n(x)\, dx$ with these distributions converge in distribution to a limit $X \sim f(x)$? Prove it.