Uniform Integrability

Let $Y \geq 0$ be integrable on some probability space $(\Omega, \mathcal{F}, \mathsf{P})$,

$$\mathsf{E}[Y] = \int_\Omega Y\, dP < \infty;$$

it follows that

$$\lim_{t\to\infty} \mathsf{E}[Y\, 1_{[Y>t]}] = \int_{[\omega: Y(\omega)>t]} Y\, dP = 0$$

and, consequently, that for any sequence of random variables X_n "dominated" by Y in the sense that $|X_n| \leq Y$ *a.s.*,

$$\begin{aligned}\lim_{t\to\infty} \mathsf{E}[X_n\, 1_{[X_n>t]}] &= \int_{[\omega: X_n(\omega)>t]} X_n\, dP \\ &\leq \int_{[\omega: Y(\omega)>t]} Y\, dP \\ &= 0, \text{ uniformly in } n.\end{aligned}$$

Call the sequence X_n *uniformly integrable* (or simply UI) if $\mathsf{E}[X_n\, 1_{[X_n>t]}] \to 0$ uniformly in n, even if it is not dominated by a single integrable random variable Y. The big result is:
Theorem: If $X_n \to X$ *in pr.* and if X_n is UI then $X_n \to X$ in L^1.
Proof. Without loss of generality take $X \equiv 0$. Fix any $\epsilon > 0$; find (by UI) $t_\epsilon > 0$ such that $\mathsf{E}[X_n\, 1_{[X_n>t_\epsilon]}] \leq \epsilon/3$ for all n. Now find (by $X_n \to X$ *in pr.*) $N_\epsilon \in \mathbb{N}$ such that, for $n \geq N_\epsilon$, $\mathsf{P}[|X_n| > \epsilon/3] < \epsilon/3t_\epsilon$; then:

$$\begin{aligned}E[|X_n|] &= \int_{[|X_n|\leq\epsilon/3]} |X_n|\, d\mathsf{P} + \int_{[\epsilon/3<|X_n|\leq t_\epsilon]} |X_n|\, d\mathsf{P} + \int_{[t_\epsilon<|X_n|]} |X_n|\, d\mathsf{P} \\ &\leq \int_{[|X_n|\leq\epsilon/3]} \epsilon/3\, d\mathsf{P} + \int_{[\epsilon/3<|X_n|\leq t_\epsilon]} t_\epsilon\, d\mathsf{P} + \int_{[t_\epsilon<|X_n|]} |X_n|\, d\mathsf{P} \\ &\leq (\epsilon/3) + (t_\epsilon)\mathsf{P}[|X_n| > \epsilon/3] + \epsilon/3 \\ &\leq \epsilon\end{aligned}$$

□

Similarly, for any $p > 0$, $X_n \to X$ (*in pr.*) and $|X_n|^p$ UI (for example, $|X_n| \leq Y \in L^p$) gives $X_n \to X$ (L^p). In the special case of $|X_n| \leq Y \in L^p$ this is called Lebesgue's **Dominated Convergence Theorem** (LDCT); Lebesgue also gave us the **Monotone Convergence Theorem** (LMCT),
Theorem: If $X_n \to X$ *in pr.* and $0 \leq X_n \nearrow X$ then $X_n \to X$ in L^1.
Proof. Without loss of generality take $\mathsf{E}X < \infty$. Since $X_n \leq X$ *a.s.*, obviously $\mathsf{E}X_n \leq \mathsf{E}X$; it remains to show for each $\epsilon > 0$ that eventually $\mathsf{E}X_n > \mathsf{E}X - \epsilon$. Fix any $\epsilon > 0$ and find $t_\epsilon > 0$ such that $\mathsf{E}[X\, 1_{[X>t_\epsilon]}] \leq \epsilon/3$. Set $\delta = \epsilon/3t_\epsilon$, and find $N_\epsilon \in \mathbb{N}$ s.t. $\forall n \geq N_\epsilon$,

$$P[A_n] < \delta; \qquad A_n \equiv \{\omega : X_n(\omega) < X(\omega) - \epsilon/3\}$$

(always possible, since $\mathsf{P}[\cap A_n] = 0$). Now fix any $n \geq N_\epsilon$; it follows that

$$
\begin{aligned}
\mathsf{E}X_n &\geq \mathsf{E}\big[(X - \epsilon/3)1_{A_n^c}\big] \\
&\geq \mathsf{E}X - \epsilon/3 - \mathsf{E}X1_{A_n} \\
&= \mathsf{E}X - \epsilon/3 - \mathsf{E}X1_{A_n}\,1_{[X \leq t_\epsilon]} - \mathsf{E}X1_{A_n}\,1_{[X > t_\epsilon]} \\
&\geq \mathsf{E}X - \epsilon/3 - t_\epsilon \mathsf{P}[A_n] - \mathsf{E}X1_{[X > t_\epsilon]} \\
&\geq \mathsf{E}X - \epsilon/3 - \epsilon/3 - \epsilon/3 \\
&= \mathsf{E}X - \epsilon
\end{aligned}
$$

$\square$

EXAMPLES

Fix any $\alpha > 0$ and define a sequence of random variables on the unit interval $(\Omega, \mathcal{F}, \mathsf{P})$ with $\Omega = (0, 1]$, $\mathcal{F} = \mathcal{B}^1$, and P Lebesgue measure (length) $\mathsf{P}(a, b] = b - a$, by

$$
X_n(\omega) = \begin{cases} \alpha^n & \text{if } 0 < \omega \leq 2^{-n} \\ 0 & \text{otherwise.} \end{cases}
$$

Then $X_n \to 0$ (*in pr.*), since $\mathsf{P}[|X_n| > 0] = 2^{-n} \to 0$, but

$$
\mathsf{E}[|X_n|^p] = (\alpha^{np})(2^{-n}) = (\alpha^p/2)^m
$$

only converges to zero if $(\alpha^p/2) < 1$, *i.e.*, $p < \log(2)/\log(\alpha)$; in particular, for $p = 1$, $X_n = \alpha^n\,1_{[\omega < 2^{-n}]}$ converges to zero (*in pr.*) but only converges to zero in L^1 if $\alpha < 2$, *i.e.*, $\beta = \frac{\log(\alpha)}{\log(2)} < 1$; in this case $|X_n| \leq Y(\omega) = \omega^{-\beta} \in L^1$. For $\alpha = 2$ we have $\mathsf{E}[X_n] = 1$ for all n, even though $X_n \to 0$ (*a.s.*).