This measure on $\mathscr{F}$ is the required extension, because by (3.7) it agrees with P on $\mathscr{F}_0$. ■

Uniqueness and the π-λ Theorem

To prove the extension in Theorem 3.1 is unique requires some auxiliary concepts. A class $\mathscr{P}$ of subsets of Ω is a *π-system* if it is closed under the formation of finite intersections:

(π) $A, B \in \mathscr{P}$ implies $A \cap B \in \mathscr{P}$.

A class $\mathscr{L}$ is a *λ-system* if it contains Ω and is closed under the formation of complements and of finite and countable *disjoint* unions:

(λ_1) $\Omega \in \mathscr{L}$;
(λ_2) $A \in \mathscr{L}$ implies $A^c \in \mathscr{L}$;
(λ_3) $A_1, A_2, \ldots, \in \mathscr{L}$ and $A_n \cap A_m = \varnothing$ for $m \neq n$ imply $\bigcup_n A_n \in \mathscr{L}$.

Because of the disjointness condition in (λ_3), the definition of λ-system is weaker (more inclusive) than that of σ-field. In the presence of (λ_1) and (λ_2), which imply $\varnothing \in \mathscr{L}$, the countably infinite case of (λ_3) implies the finite one.

In the presence of (λ_1) and (λ_3), (λ_2) is equivalent to the condition that $\mathscr{L}$ is closed under the formation of proper differences:

(λ'_2) $A, B \in \mathscr{L}$ and $A \subset B$ imply $B - A \in \mathscr{L}$.

Suppose, in fact, that $\mathscr{L}$ satisfies (λ_2) and (λ_3). If $A, B \in \mathscr{L}$ and $A \subset B$, then $\mathscr{L}$ contains B^c, the disjoint union $A \cup B^c$, and its complement $(A \cup B^c)^c = B - A$. Hence (λ'_2). On the other hand, if $\mathscr{L}$ satisfies (λ_1) and (λ'_2), then $A \in \mathscr{L}$ implies $A^c = \Omega - A \in \mathscr{L}$. Hence ($\lambda_2$).

Although a σ-field is a λ-system, the reverse is not true (in a four-point space take $\mathscr{L}$ to consist of $\varnothing$, Ω, and the six two-point sets). But the connection is close:

Lemma 6. *A class that is both a π-system and a λ-system is a σ-field.*

PROOF. The class contains Ω by (λ_1) and is closed under the formation of complements and finite intersections by (λ_2) and (π). It is therefore a field. It is a σ-field because if it contains sets A_n, then it also contains the disjoint sets $B_n = A_n \cap A_1^c \cap \cdots \cap A_{n-1}^c$ and by (λ_3) contains $\bigcup_n A_n = \bigcup_n B_n$. ■

Many uniqueness arguments depend on *Dynkin's π-λ theorem*:

Theorem 3.2. *If $\mathscr{P}$ is a π-system and $\mathscr{L}$ is a λ-system, then $\mathscr{P} \subset \mathscr{L}$ implies $\sigma(\mathscr{P}) \subset \mathscr{L}$.*

PROOF. Let $\mathscr{L}_0$ be the λ-system generated by $\mathscr{P}$—that is, the intersection of all λ-systems containing $\mathscr{P}$. It is a λ-system, it contains $\mathscr{P}$, and it is contained in every λ-system that contains $\mathscr{P}$ (see the construction of generated σ-fields, p. 21). Thus $\mathscr{P} \subset \mathscr{L}_0 \subset \mathscr{L}$. If it can be shown that $\mathscr{L}_0$ is also a π-system, then it will follow by Lemma 6 that it is a σ-field. From the minimality of $\sigma(\mathscr{P})$ it will then follow that $\sigma(\mathscr{P}) \subset \mathscr{L}_0$, so that $\mathscr{P} \subset \sigma(\mathscr{P}) \subset \mathscr{L}_0 \subset \mathscr{L}$. Therefore, it suffices to show that $\mathscr{L}_0$ is a π-system.

For each A, let $\mathscr{L}_A$ be the class of sets B such that $A \cap B \in \mathscr{L}_0$. If A is assumed to lie in $\mathscr{P}$, or even if A is merely assumed to lie in $\mathscr{L}_0$, then $\mathscr{L}_A$ *is a λ-system*: Since $A \cap \Omega = A \in \mathscr{L}_0$ by the assumption, $\mathscr{L}_A$ satisfies (λ_1). If $B_1, B_2 \in \mathscr{L}_A$ and $B_1 \subset B_2$, then the λ-system $\mathscr{L}_0$ contains $A \cap B_1$ and $A \cap B_2$ and hence contains the proper difference $(A \cap B_2) - (A \cap B_1) = A \cap (B_2 - B_1)$, so that $\mathscr{L}_A$ contains $B_2 - B_1$: $\mathscr{L}_A$ satisfies (λ'_2). If B_n are disjoint $\mathscr{L}_A$-sets, then $\mathscr{L}_0$ contains the disjoint sets $A \cap B_n$ and hence contains their union $A \cap (\bigcup_n B_n)$: $\mathscr{L}_A$ satisfies (λ_3).

If $A \in \mathscr{P}$ and $B \in \mathscr{P}$, then ($\mathscr{P}$ is a π-system) $A \cap B \in \mathscr{P} \subset \mathscr{L}_0$, or $B \in \mathscr{L}_A$. Thus $A \in \mathscr{P}$ implies $\mathscr{P} \subset \mathscr{L}_A$, and since $\mathscr{L}_A$ is a λ-system, minimality gives $\mathscr{L}_0 \subset \mathscr{L}_A$.

Thus $A \in \mathscr{P}$ implies $\mathscr{L}_0 \subset \mathscr{L}_A$, or, to put it another way, $A \in \mathscr{P}$ and $B \in \mathscr{L}_0$ together imply that $B \in \mathscr{L}_A$ and hence $A \in \mathscr{L}_B$. (The key to the proof is that $B \in \mathscr{L}_A$ if and only if $A \in \mathscr{L}_B$.) This last implication means that $B \in \mathscr{L}_0$ implies $\mathscr{P} \subset \mathscr{L}_B$. Since $\mathscr{L}_B$ is a λ-system, it follows by minimality once again that $B \in \mathscr{L}_0$ implies $\mathscr{L}_0 \subset \mathscr{L}_B$. Finally, $B \in \mathscr{L}_0$ and $C \in \mathscr{L}_0$ together imply $C \in \mathscr{L}_B$, or $B \cap C \in \mathscr{L}_0$. Therefore, $\mathscr{L}_0$ is indeed a π-system. ■

Since a field is certainly a π-system, the uniqueness asserted in Theorem 3.1 is a consequence of this result:

Theorem 3.3. *Suppose that P_1 and P_2 are probability measures on $\sigma(\mathscr{P})$, where $\mathscr{P}$ is a π-system. If P_1 and P_2 agree on $\mathscr{P}$, then they agree on $\sigma(\mathscr{P})$.*

PROOF. Let $\mathscr{L}$ be the class of sets A in $\sigma(\mathscr{P})$ such that $P_1(A) = P_2(A)$. Clearly $\Omega \in \mathscr{L}$. If $A \in \mathscr{L}$, then $P_1(A^c) = 1 - P_1(A) = 1 - P_2(A) = P_2(A^c)$, and hence $A^c \in \mathscr{L}$. If A_n are disjoint sets in $\mathscr{L}$, then $P_1(\bigcup_n A_n) = \sum_n P_1(A_n) = \sum_n P_2(A_n) = P_2(\bigcup_n A_n)$, and hence $\bigcup_n A_n \in \mathscr{L}$. Therefore $\mathscr{L}$ is a λ-system. Since by hypothesis $\mathscr{P} \subset \mathscr{L}$ and $\mathscr{P}$ is a π-system, the π-λ theorem gives $\sigma(\mathscr{P}) \subset \mathscr{L}$, as required. ■

Note that the π-λ theorem and the concept of λ-system are exactly what are needed to make this proof work: The essential property of probability measures is countable additivity, and this is a condition on countable *disjoint* unions, the only kind involved in the requirement (λ_3) in the definition of λ-system. In this, as in many applications of the π-λ theorem, $\mathscr{L} \subset \sigma(\mathscr{P})$ and therefore $\sigma(\mathscr{P}) = \mathscr{L}$, even though the relation $\sigma(\mathscr{P}) \subset \mathscr{L}$ itself suffices for the conclusion of the theorem.

Monotone Classes

A class $\mathscr{M}$ of subsets of Ω is *monotone* if it is closed under the formation of monotone unions and intersections:

(i) $A_1, A_2, \ldots \in \mathscr{M}$ and $A_n \uparrow A$ imply $A \in \mathscr{M}$;
(ii) $A_1, A_2, \ldots \in \mathscr{M}$ and $A_n \downarrow A$ imply $A \in \mathscr{M}$.

Halmos's monotone class theorem is a close relative of the π-λ theorem but will be less frequently used in this book.

Theorem 3.4. *If $\mathscr{F}_0$ is a field and $\mathscr{M}$ is a monotone class, then $\mathscr{F}_0 \subset \mathscr{M}$ implies $\sigma(\mathscr{F}_0) \subset \mathscr{M}$.*

PROOF. Let $m(\mathscr{F}_0)$ be the minimal monotone class over $\mathscr{F}_0$—the intersection of all monotone classes containing $\mathscr{F}_0$. It is enough to prove $\sigma(\mathscr{F}_0) \subset m(\mathscr{F}_0)$; this will follow if $m(\mathscr{F}_0)$ is shown to be a field, because a monotone field is a σ-field.

Consider the class $\mathscr{G} = [A: A^c \in m(\mathscr{F}_0)]$. Since $m(\mathscr{F}_0)$ is monotone, so is $\mathscr{G}$. Since $\mathscr{F}_0$ is a field, $\mathscr{F}_0 \subset \mathscr{G}$, and so $m(\mathscr{F}_0) \subset \mathscr{G}$. Hence $m(\mathscr{F}_0)$ is closed under complementation.

Define $\mathscr{G}_1$ as the class of A such that $A \cup B \in m(\mathscr{F}_0)$ for all $B \in \mathscr{F}_0$. Then $\mathscr{G}_1$ is a monotone class and $\mathscr{F}_0 \subset \mathscr{G}_1$; from the minimality of $m(\mathscr{F}_0)$ follows $m(\mathscr{F}_0) \subset \mathscr{G}_1$. Define $\mathscr{G}_2$ as the class of B such that $A \cup B \in m(\mathscr{F}_0)$ for all $A \in m(\mathscr{F}_0)$. Then $\mathscr{G}_2$ is a monotone class. Now from $m(\mathscr{F}_0) \subset \mathscr{G}_1$ it follows that $A \in m(\mathscr{F}_0)$ and $B \in \mathscr{F}_0$ together imply that $A \cup B \in m(\mathscr{F}_0)$; in other words, $B \in \mathscr{F}_0$ implies that $B \in \mathscr{G}_2$. Thus $\mathscr{F}_0 \subset \mathscr{G}_2$; by minimality, $m(\mathscr{F}_0) \subset \mathscr{G}_2$, and hence $A, B \in m(\mathscr{F}_0)$ implies that $A \cup B \in m(\mathscr{F}_0)$. ∎

Lebesgue Measure on the Unit Interval

Consider once again the unit interval $(0, 1]$ together with the field $\mathscr{B}_0$ of finite disjoint unions of subintervals (Example 2.2) and the σ-field $\mathscr{B} = \sigma(\mathscr{B}_0)$ of Borel sets in $(0, 1]$. According to Theorem 2.2, (2.12) defines a probability measure λ on $\mathscr{B}_0$. By Theorem 3.1, λ extends to $\mathscr{B}$, the extended λ being Lebesgue measure. The probability space $((0, 1], \mathscr{B}, \lambda)$ will be the basis for much of the probability theory in the remaining sections of this chapter. A few geometric properties of λ will be considered here. Since the intervals in $(0, 1]$ form a π-system generating $\mathscr{B}$, λ is the only probability measure on $\mathscr{B}$ that assigns to each interval its length as its measure.