Final Examination

STA 215: Statistical Inference

Wednesday, 2000 May 3, 7:00 – 10:00 pm

This is an open-book examination, but you may not share materials. A normal distribution table, the PDF handout, and a blank worksheet are attached to the exam. If you don't understand something in one of the questions feel free to ask me, but please do not talk to each other. You may use a calculator but not a laptop computer.

You must **show** your **work** to get partial credit. Unsupported answers are not acceptable, even if they are correct. It is to your advantage to write your solutions as clearly as possible, since I cannot give credit for solutions I do not understand. Good luck.

1.	/20
2.	/20
3.	/20
4.	/20
5.	/20
X.	/00
Total:	/100

Problem 1: Let $X_j \sim \mathsf{Ga}(a, \lambda)$ be independent gamma-distributed random variables with **known** shape parameter a, each with density function

$$f(x \mid a, \lambda) = \frac{\lambda^a \, x^{a-1}}{\Gamma(a)}, e^{-\lambda x}, \qquad x > 0, \tag{1}$$

and answer the following questions.

a. (10) Find the posterior distribution for λ and the posterior mean $\mathsf{E}[\lambda \mid X]$ for n observations $X = [x_1, \ldots, x_n]$, using a Gamma prior distribution $\lambda \sim \mathsf{Ga}(\alpha, \theta)$.

b. (10) With a known, find the Jeffreys' prior distribution $\pi_J(d\lambda)$, the posterior distribution for λ with this prior, and the posterior mean $\mathsf{E}_J[\lambda \mid X]$.

Problem 2: With the same Gamma model as above, again for fixed $a > 0$,

a. (10) Find the maximum likelihood estimator $\hat{\lambda}$ on the basis of a random sample of size n.

b. (10) Perform a significance test of the hypothesis $H_0 : \lambda \leq 1$ against the alternative $H_1 : \lambda > 1$. Give either an `S-Plus` command or a (one-dimensional) integral expression for the P-value, as a function of a, n, and $\mathbf{x} = (x_1, ..., x_n)$.

Problem 3: The Pareto distribution with density function

$$f(x \mid \alpha, \beta) = \beta\alpha^\beta / x^{\beta+1}, \qquad x \in [\alpha, \infty)$$

for $\alpha > 0$, $\beta > 0$ is sometimes used to model incomes or other positive quantities with "fat" tails.

a) (4) Find the likelihood function $f_n(\mathbf{x} \mid \alpha, \beta)$ on the basis of n independent observations $\mathbf{x} = (x_1, ..., x_n)$. Be careful.

b) (4) Find the maximum likelihood estimates $\hat{\alpha}_n$, $\hat{\beta}_n$ upon observing $\mathbf{x} = (x_1, ..., x_n)$.

c) (4) Find a (possibly vector-valued) sufficient statistic $S(\mathbf{x}) = (S_1(\mathbf{x}), ..., S_k(\mathbf{x}))$ for $\theta = (\alpha, \beta)$, for $\mathbf{x} = (x_1, ..., x_n)$.

d) (8) Give the posterior distribution for β, if α is known (i.e. has a point-mass prior) and if β has a reference prior distribution $\pi(d\beta) = \beta^{-1}\, d\beta$.

Problem 4: The random variables $X_i \sim \mathsf{No}(1, \sigma^2)$ are drawn independently from a normal distribution with known mean $\mu = 1$ and unknown variance $\sigma^2 > 0$.

a) (5) If the precision (inverse variance) σ^{-2} is accorded a Gamma prior distribution $\sigma^{-2} \sim \mathsf{Ga}(\alpha, \lambda)$, find the posterior distribution for σ^{-2} upon observing $\mathbf{x} = (x_1, ..., x_n)$.

b) (10) Find the Jeffreys prior density $\pi_J(\sigma^{-2})$ for the precision.

c) (5) Give the posterior mean $\mathsf{E}[\sigma^2 \mid \mathbf{x}]$ for the *variance*, using the Gamma prior of a) above. (Hint: Recall that $Y \sim \mathsf{Ga}(\alpha, \lambda) \Rightarrow \mathsf{E}[Y^p] = \Gamma(\alpha + p)\lambda^{-p}/\Gamma(\alpha)$).

Problem 5: Ron thinks that the recorded failure times t_j of felt dry-board marking pens have a $\mathsf{Ga}(2,2)$ distribution, with mean one hour, while Tom thinks they have an $\mathsf{Ex}(1)$ distribution, also with mean one hour. Consider the following statistics, each based on a sample of size n:

$S_1 = \sum_{j=1}^n t_j$	$S_2 = \sum_{j=1}^n {t_j}^2$	$S_3 = \sum_{j=1}^n \log(t_j)$
$S_4 = \min_{1 \le j \le n} t_j$	$S_5 = \max_{1 \le j \le n} t_j$	$S_6 = \sum_{j=1}^n 1/t_j$

(20) How should Ron and Tom resolve their dispute? Suggest and defend a statistic based on one or more of the $S_k = S_k(\mathbf{t}_n)$ above and a procedure for deciding whether $t_j \sim \mathsf{Ga}(2,2)$ or $t_j \sim \mathsf{Ex}(1)$. Describe briefly what computation would be required. Please present **both** a (10) Frequentist and a (10) Bayesian solution.

Problem X: Extra Credit:

a) (+5) (Problem 1 b., revisited). If $X_j \sim \mathsf{Ga}(\alpha, \lambda)$ with both parameters unknown, find the (joint) Jeffreys prior density $\pi_J(\alpha, \lambda)$ for the two parameters. (Notational hint: Define $\gamma(z) \equiv \log \Gamma(z)$ for $z > 0$, with derivatives $\psi(z) \equiv \gamma'(z)$, called the *digamma function* and $\psi'(z) \equiv \gamma''(z)$, the *trigamma function*).

b) (+5) (Problem 5, revisited). Find a normal approximation to (any function of) the statistic you proposed in problem 5 and, from this, give an approximate but explicit rule for deciding between Ron and Tom's claims.

c) (+1) What are Ron and Tom's *last* names?

Extra worksheet, if needed:

Name: ________________

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\, e^{-t^2/2}\, dt:$$

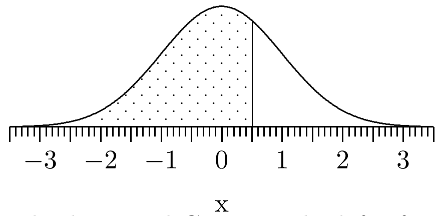

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998