STAT215: Homework 3

Due: Monday, March 6

1. (20 pt) For each of the following distributions, let $X_1, \cdots, X_n$ be a random sample. Is there a function of θ, say $g(\theta)$, for which there exists an unbiased estimator whose variance attains the Cramér-Rao Lower Bound? If so, find it. If not, show why not.

   (a) $f(x|\theta) = \theta x^{\theta-1}, \ \ 0 < x < 1, \ \theta > 0$

   (b) $f(x|\theta) = \frac{\log(\theta)}{\theta-1}\theta^x, \ \ 0 < x < 1, \ \theta > 1$

2. (15 pt) Let $X_1, \cdots, X_n$ be iid Bernoulli(p). Show that the variance of $\bar{X}$ attains the Cramér-Rao Lower Bound, and hence $\bar{X}$ is the best unbiased estimator of p.

3. (15 pt) Let $X_1, \cdots, X_n$ be a random sample from a population with mean μ and variance σ^2.

   (a) Show that the estimator $\sum_{i=1}^n a_i X_i$ is an unbiased estimator of μ if $\sum_{i=1}^n a_i = 1$.

   (b) Among all unbiased estimators of this form (called *linear unbiased estimators*) find the one with minimum variance, and calculate the variance.

4. (20 pt) Let $X_1, \cdots, X_n$ be iid $\mathsf{N}(\theta, 1)$. Show that the best unbiased estimator of θ^2 is $\bar{X}^2 - (1/n)$. Calculate its variance and show that it is greater than the Cramér-Rao Lower Bound.

   (Hint1: use Stein's Identity Let $X \sim \mathsf{N}(\theta, \sigma^2)$, and let g be a differentiable function satisfying $\mathbf{E}[|g^{'}(X)|] < \infty$. Then $\mathbf{E}[g(X)(X-\theta)] = \sigma^2 \mathbf{E}[g^{'}(X)]$.)

   (Hint2: ust Theorem: Let T be a complete sufficient statistic for a parameter θ, and let $\phi(T)$ be any estimator based only on T. Then $\phi(T)$ is the unique best unbiased estimator of its expected value. In normal case, $\bar{X}$ is complete and sufficient for the mean when variance is known.)

5. (30pt) Let X_j be a sequence of independent $\mathsf{Po}(\lambda)$ random variables.

   (1). Find the MLE $\hat{\lambda}_n$ on the basis of the first n observations and show $\hat{\lambda}_n$ is sufficient for λ

   (2) Find the Fisher information $I(\lambda)$.

   (3) On the basis of the $n = 6$ observations $\mathbf{x} = \{1, 0, 2, 4, 3, 0\}$, find the 10% Likelihoodist Interval for λ (i.e., the set of points λ where the LH function attains at least 10% of its maximum value).

   (4) On the basis of the same $n = 6$ observations as before, find the equatail 90% Confidence Interval for λ, correct to four decimal places. Show the code in your choice of S-Plus, R or Matlab needed for your answer.

   (5) Find the Jeffery's prior distribution for λ, and, on the basis of the same $n = 6$ observations as before, find the Bayesian posterior distribution and the posterior mean $\bar{\lambda}_n = \mathbf{E}[\lambda|\mathbf{x}]$.

   (6) Find the Bayesian equal-tail 90% Credible Interval for λ, using the Jeffery's prior.

6. (10 pt) BONUS question

   Find the Jeffery's Prior for a location-scale family, i.e. one where $f(x|a,b) = b * f((x-a)*b)$ for $-\infty < a < \infty, \ 0 < b < \infty$ for some pdf $f(z)$. (Hint: Write $f(z) = e^{-\phi(z)}$ and do everything in terms of $\phi(z) \equiv -\log(z)$; change variables in the expectation step to $z = b(x-a)$.)