The curriculum for this statistics class assumes students are familiar with linear algebra, vector calculus, and the material from a calculus-based probability class. The problems on this page are typical of the kind of calculations we will encounter throughut the course. You should see how many of them you can do without help. Your probability text will contain the material needed to do them; solutions are given on the back of this page. Common distributions' pdfs and pmfs ($\mathsf{Po}(\lambda)$, $\mathsf{Ge}(p)$, etc.) are available on the course PDF page.

1. Find the mean μ_n, variance σ_n^2, and exact distribution (name and parameters) of $S_n = \sum_{i=1}^n X_i$ if $\{X_i\}$ are iid with the:
   a) $\mathsf{Po}(\lambda)$ b) $\mathsf{Ge}(p)$ c) $\mathsf{Ex}(\lambda)$ d) $\mathsf{No}(\mu, \sigma^2)$ e) $\mathsf{NB}(\alpha, p)$ f) $\mathsf{Ga}(\alpha, \lambda)$
   distributions.
2. Find $\mathsf{E}[\sin(U)]$ for $U \sim \mathsf{Un}(0, \pi)$.
3. If $X \sim \mathsf{Ga}(\alpha, \lambda)$, find $\mathsf{E}[X^p]$ for every $p \in \mathbb{R}$.
4. What is meant by the sum of the infinite series $1 + x + x^2 + x^3 + \cdots$? What is the sum (in terms of x)? How would you show that your answer is correct?
5. Find the sum of the series $(4/5)^3 + (4/5)^4 + (4/5)^5 + \cdots + (4/5)^{20}$.
6. What is the sum of the series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$? For which x does it converge?
7. If $X \sim \mathsf{Po}(\lambda)$ and $Y \sim \mathsf{Ge}(p)$ are independent, find $\mathsf{P}[X \le Y]$.
8. If $X, Y \overset{\text{iid}}{\sim} \mathsf{No}(\mu, \sigma^2)$, are $(X - Y)$ and $(X + Y)$ independent?
9. Find the logarithm of $\frac{a\ b^c}{d \exp(x)}$. Simplify.
10. Find (a) $\lim_{n\to\infty}(1 - 2/n^2)^{n^2}$ and (b) $\lim_{n\to\infty}(a + b\,n^2)/(c\,n + d\,n^2)$.
11. Sketch the triangle in the plain whose vertices are at $(0, 0)$, $(0, 1)$ and $(1, 0)$. Find the integral of $f(x, y) = x + xy$ over that triangle.
12. Find the integral of $f(x, y) = e^{-(x^2+y^2)}$ over the first quadrant in the plane (Hint: Use polar coordinates).

Solutions

1. Means $n\lambda$, nq/p, n/λ, $n\mu$, $n\alpha q/p$, $n\alpha/\lambda$; variances $n\lambda$, nq/p^2, n/λ^2, $n\sigma^2$, $n\alpha q/p^2$, $n\alpha/\lambda^2$; distns $\mathsf{Po}(n\lambda)$, $\mathsf{NB}(n,p)$, $\mathsf{Ga}(n,\lambda)$, $\mathsf{No}(n\mu, n\sigma^2)$, $\mathsf{NB}(n\alpha,p)$, $\mathsf{Ga}(n\alpha,\lambda)$.

2. $\mathsf{E}\sin(U) = \frac{1}{\pi}\int_0^\pi \sin(x)\,dx = \frac{-1}{\pi}\cos(x)|_{x=0}^{x=\pi} = \frac{2}{\pi}$.

3. $\frac{\lambda^\alpha}{\Gamma(\alpha)}\int_0^\infty x^p\, x^{\alpha-1}e^{-\lambda x}\,dx = \frac{\lambda^\alpha}{\Gamma(\alpha)}\lambda^{-p-\alpha}\int_0^\infty y^{p+\alpha-1}\,e^{-y}\,dy = \frac{\Gamma(\alpha+p)}{\Gamma(\alpha)\lambda^p}$ if $p > -\alpha$, ∞ if $p \le -\alpha$.

4. Call the partial sum of this Geometric Series $S_n = 1 + x + x^2 + x^3 + \cdots + x^n$; the infinite series is defined to be the limit $S = \lim_{n\to\infty} S_n$. It's easy to show that the finite sum is $S_n = (1 - x^{n+1})/(1-x)$, if $x \ne 1$, or $S_n = n$, if $x = 1$ (verify by multiplying both sides by $(1-x)$ and cancelling the "telescoping series"), so the infinite sum is $S = \lim_{n\to\infty}(1 - x^{n+1})/(1-x) = 1/(1-x)$ if $|x| < 1$. The limit is infinite for $x \ge 1$ and undefined for $x \le -1$. The way to remember it is: *First term included, minus first term omitted, divided by one minus the ratio.* This is a good one to memorize.

5. By the recipe above, it's $\left[(4/5)^3 - (4/5)^{21}\right]/(1-4/5) = 2.56 - 0.04611686 = 2.513883$.

6. The exponential series $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots = e^x$ converges for all real numbers x.

7. $\mathsf{P}[X \le Y] = \sum_{k=0}^\infty \mathsf{P}[X = k,\ Y \ge k] = \sum_0^\infty \frac{\lambda^k}{k!}e^{-\lambda}\,q^k = e^{\lambda q - \lambda} = e^{-\lambda p}$, where $q \equiv 1 - p$.

8. $(X-Y)$ and $(X+Y)$ have means 0 and 2μ respectively, each with variance $2\sigma^2$. Since they are jointly normal and have zero covariance (since $E[X^2 - Y^2] = 0$), they *are* independent.

9. $\log a + c\log b - \log d - x$.

10. $\lim_{m\to\infty}(1 + x/m)^m = e^x$ for any (real or complex) x; in (a) we have $x = -2$ and $m = n^2$, so the limit is e^{-2}. For (b), either l'Hôspital's rule or dividing numerator and denominator by n^2 will lead to the same limit, b/d, if $d \ne 0$. Limit is $\pm\infty$ if $d = 0$, $c \ne 0$, and $\pm b/c > 0$; 0 if $b = 0$ and $c \ne 0$; and undefined if $c = d = 0$. Did you get them all?

11. The double integral can be set up in either of two ways, integrating first with respect to x or to y. If we choose to integrate over horizontal lines (first dx, then dy) we find that for each $0 \le y \le 1$ the range of x in the triangle is $0 \le x \le 1 - y$, so

$$\iint_\Delta f(x,y)dx\,dy = \int_0^1\left[\int_0^{1-y}(x + xy)\,dx\right]dy = \int_0^1\left[(1+y)(1-y)^2/2\right]dy = 5/24$$

If instead we choose to integrate over vertical lines (first dy, then dx) we find that for each $0 \le x \le 1$ the range of y in the triangle is $0 \le y \le 1 - x$, so

$$\iint_\Delta f(x,y)dy\,dx = \int_0^1\left[\int_0^{1-x}(x + xy)\,dy\right]dx = \int_0^1\left[x(1-x) + x(1-x)^2/2\right]dx = 5/24.$$

12. Switch to polar coordinates, then change variables from r to $t \equiv r^2$:

$$\int_0^\infty\!\!\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty e^{-r^2}r\,dr\,d\theta = (1/2)(\pi/2) = \pi/4.$$