Final Examination

Mth 135 = Sta 104

Wednesday, 2010 December 15, 2:00 – 5:00pm

- This is a **closed book** exam— put your books & notes on the floor.
- You may use a calculator and **two pages** of your own notes.
- Do not share calculators or notes. **No phones** or other network-connected devices may be used as calculators or timers.
- Please **ask me** questions if a problem needs clarification.
- **Show** your **work**. Neatness counts. Boxing answers helps.
- Numerical answers: **four sig**nificant **dig**its or fractions **in lowest terms**. Simplify *all* expressions.
- Extra worksheet and pdf & normal distribution tables are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Signature: ______________________

Print Name: ______________________

1.	/20	5.	/20
2.	/20	6.	/20
3.	/20	7.	/20
4.	/20	8.	/20
Total:			/160

Problem 1: Answer the following questions about the events A, B, C:

a) If A and B are independent and each has probability 2/3, find:

$\mathsf{P}[A^c \cup B] =$ ____________________

b) If $\mathsf{P}[A \mid B] = 1/3$ and $\mathsf{P}[B \mid A] = 1/3$, what is the largest possible value for $\mathsf{P}[A \cap B]$? Find the least upper bound:

$\mathsf{P}[A \cap B] \leq$ ____________________

c) If $\mathsf{P}[A \cup B] = 2/3$ and $\mathsf{P}[A] = \mathsf{P}[B] = 1/2$, find:

$\mathsf{P}[A \mid B] =$ ____________________

d) If A, B, C are independent and each has probability 1/4, find:

$\mathsf{P}[A \cup B \cup C] =$ ____________________

e) The events A and B are disjoint. The probability that B will not happen is 0.60, and $\mathsf{P}[A \cup B] = 0.5$. Find:

$\mathsf{P}[A] =$ ____________________

Problem 2: Two coins look similar, but have different probabilities of falling "heads". One is a **Fair** coin, with $\mathsf{P}[H] = 1/2$, but the other is weighted so that $\mathsf{P}[H] = 8/10$. One of the coins is chosen at random and is tossed 10 times; **X** is the number of Heads to appear, and **F** is the event that the **fair** coin was drawn. As always, **simplify** your answers

a) If $X = 7$ is observed, what is the probability that the coin was fair?

$\mathsf{P}[F \mid X = 7] =$ ____________

b) If $X \geq 9$, what is the probability that $X = 10$?

$\mathsf{P}[X = 10 \mid X \geq 9] =$ ____________

Problem 3: Good news! You get a box of one dozen chocolates, of three kinds: six Plain, four Coconut, and two Cashew (they all look alike). Draw them one at a time at random and eat them; obviously draws are made *without* replacement.

a) What is the chance the first three are all *the same* type?

b) What is the chance the first three are all of *different* types?

c) What is the probability that the first Cashew will be found on the fifth draw?

d) Find the expected number of Plain chocolates eaten before any Cashew:

Problem 4: Let X, Y have joint pdf

$$f(x,y) = \frac{1}{2}(x+y)\,e^{-(x+y)} \qquad 0 < x < \infty,\ 0 < y < \infty.$$

a) (10) Find the probability density function $f_Z(z)$ for the sum $Z = X + Y$, correctly for all $-\infty < z < \infty$:
$f_Z(z) =$

b) (5) Find the joint probability density function $f(x,z)$ for X and Z, correctly for all $-\infty < x < \infty$ and $-\infty < z < \infty$:
$f(x,z) =$

c) (5) Find the conditional probability density function $f(x \mid z)$ for X given the sum $Z = X + Y$, correctly for all $-\infty < x < \infty$:
$f(x \mid z) =$

Problem 5: In a certain gambling game three fair coins are thrown in the air. If they match (all heads **or** all tails), you win \$$C$; if they don't, you pay \$1.

a) (7) For the game to be "fair" in the sense that your long-run average gain (or loss) will be zero, what must be the value of \$$C$?

$C =$ ____________ Why?

b) (6) Using the value of C you found above, find the mean and variance of your net winnings W_{75} after 75 independent plays:

$\mathsf{E}[W_{75}] =$ ____________ $\mathsf{Var}[W_{75}] =$ ____________

c) (7) Using the value of C you found above, use the Central Limit Theorem to find (to three correct decimal places) the approximate probability of a net loss of at least \$20 in 75 plays (2 pt bonus for *also* including unevaluated sum or integral expression for the *exact* probability):

$\mathsf{P}[W_{75} \leq -20] \approx$ ____________

Problem 6: Let X and Y be two points drawn independently from the unit interval $(0,1)$ with pdfs $f_X(x) = 2x\mathbf{1}_{\{0<x<1\}}$ and $f_Y(y) = 3y^2\mathbf{1}_{\{0<y<1\}}$, respectively. Find the:

a) Means:

$\mathsf{E}[X] =$ ____________ $\mathsf{E}[Y] =$ ____________

b) Probabilities:

$\mathsf{P}[X < 1/2] =$ ____________ $\mathsf{P}[Y < 1/2] =$ ____________

c) Probability:

$\mathsf{P}[X < Y] =$ ____________

Problem 7: Let X and Y be normally-distribute random variables

$$X \sim \mathsf{No}(2,4) \qquad Y \sim \mathsf{No}(1,25)$$

with *variances* $2^2 = 4$ and $5^2 = 25$ and with covariance

$$\mathsf{Cov}(X,Y) = 6$$

a) Find the mean and variance of $X - 2Y$:

$\mathsf{E}[X - 2Y] =$ ____________ $\mathsf{Var}[X - 2Y] =$ ____________

b) Find numbers $a,\ b,\ c,\ d,\ e$ so that we may write

$$X = a + b\,Z_1$$
$$Y = c + d\,Z_1 + e\,Z_2$$

for *independent* random variables $Z_1,\ Z_2 \sim \mathsf{No}(0,1)$ (Hint: In terms of a, b, c, d, e, what *are* the means, variances, and covariance of X, Y?)

c) Find a constant ϕ so that X and $Z = (Y - \phi X)$ are independent. Find the mean and variance of Z, too.

$\phi =$ __________ $\mu_Z =$ __________ $\sigma_Z^2 =$ __________

d) $\mathsf{P}[Y \le 7.5 \mid X = 1] =$ ____________
(Hint: Re-write Y using your answer to b) or c) above)

Problem 8: Let $X \sim \mathsf{Be}(\theta, 1)$ have a Beta $\mathsf{Be}(\alpha, \beta)$ dist'n[1] with parameters $\alpha = \theta$ and $\beta = 1$ for some real number $\theta > 0$. Give requested pdf's correctly at **all** real numbers x, y, *etc.* **Simplify**— no answer should have a $\Gamma(\cdot)$. Each answer will depend on the value of $\theta > 0$.

a) (2) $f_X(x) =$ ____________ $\mathsf{E}[X] =$ ____________
Give the probability density function (pdf) and mean for X.

b) (6) $f_Y(y) =$ ____________ $\mathsf{E}[Y] =$ ____________
Give the pdf and mean for $Y = \sqrt{X}$.

c) (6) $f_z(z) =$ ____________ $\mathsf{E}[Z] =$ ____________
Give the pdf and mean for $Z = -\log X$ (the *natural* logarithm)

d) (6) $f_R(r) =$ ____________ $\mathsf{E}[R] =$ ____________
Give the pdf and mean for $R = 1/X$.

[1] and hence density proportional to $x^{\alpha-1}(1-x)^{\beta-1}\mathbf{1}_{\{0<x<1\}}$; see pdf table on *p.* 12.

Extra worksheet, if needed:

Another extra worksheet, if needed:

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$

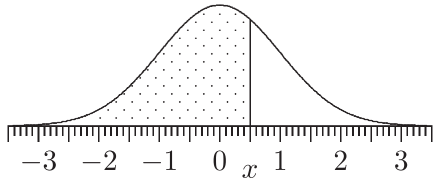

Table 5.1 Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	