

MLE and curvature

From glm() function in R

$$\hat{\beta}_{\mathsf{MLE}} = \begin{pmatrix} -3.170\\ 0.079\\ 1.064\\ -0.003 \end{pmatrix}$$

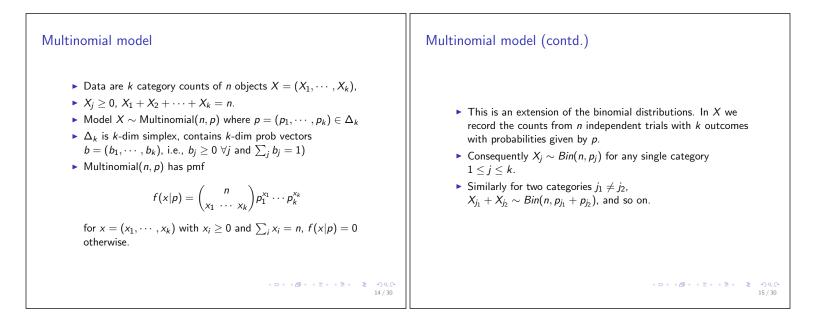
$$I_X^{-1} = \begin{pmatrix} 0.065 & -0.003 & -0.065 & 0.003 \\ -0.003 & 0.001 & 0.003 & -0.001 \\ -0.065 & 0.003 & 0.192 & -0.011 \\ 0.003 & -0.001 & -0.011 & 0.005 \end{pmatrix}$$

- How would you test the null hypothesis that for a black mother, probability of low birthweight depends on the number of cigarettes? For a non-black mother?
- How would you test that this dependence is different for black and non-black mothers?

Categorical data & Multinomial models

- Data: Counts of categories formed by one or more attributes
- Tables could be one-way, two-way, etc., depending on how many attributes are used to decide categories.

0.003			Eye color					
-0.001			Blue	•	Brown	Black	Total	
$\begin{pmatrix} -0.011 \\ 0.005 \end{pmatrix}$	 or	Blonde	20	15	18	14	67	
0.000 /	color	Red	11	4	24	2	41	
at for a black	Hair	Brown	9	11	36	18	74	
ends on the number	Ξ	Black	8	17	20	4	49	
		Total	48	47	98	38	231	
is different for black								
나 《御》《言》《言》 홈 옛역 12/30						< = > < č	9 → < ≅ →	



Multinomial model (contd.)MLE• For two-way tables, with k_1 rows and k_2 columns, we can
write X and p either as k_1k_2 -dim vectors, or more commonly
as $k_1 \times k_2$ matrices.• To obtain a
maximize t
Lagrange c
 $\sum_{j=1}^{k} p_j =$
 $\tilde{\ell}_x(x_j)$ • Even when they're written as matrices, we'll write
X ~ Multinomial(n, p) to mean that the multinomial
distribution is placed on the vector forms of X and p.• The solution• For multi-way table we'll use arrays of appropriate dimensions
to represent X and p.• The solution

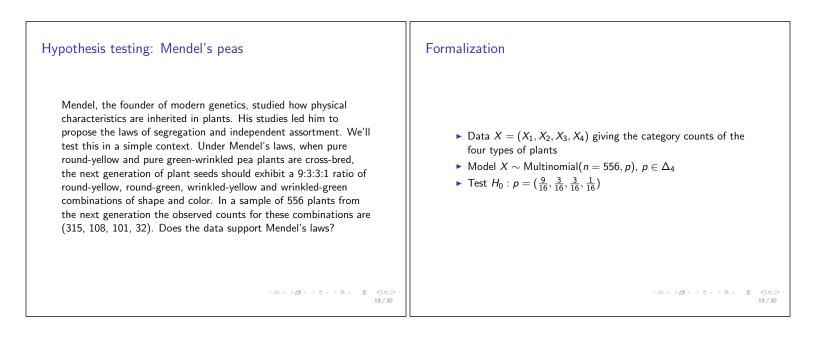
► To obtain the MLE of p based on data X = x, we can maximize the log-likelihood function in p with an additional Lagrange component to account for the constraint ∑_{i=1}^k p_i = 1:

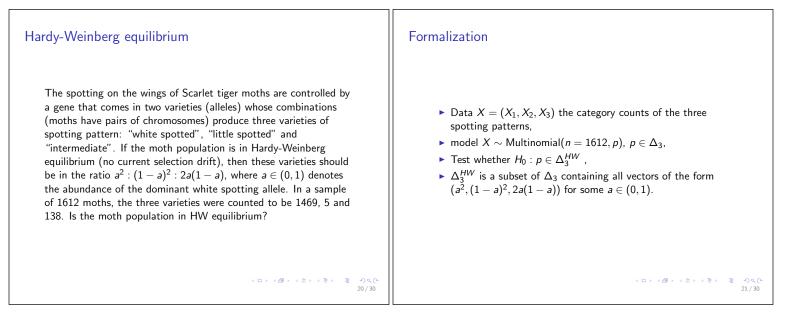
$$ilde{\ell}_x(\pmb{p},\lambda) = ext{const} + \sum_{j=1}^k x_j \log p_j + \lambda (\sum_{j=1}^k p_j - 1)$$

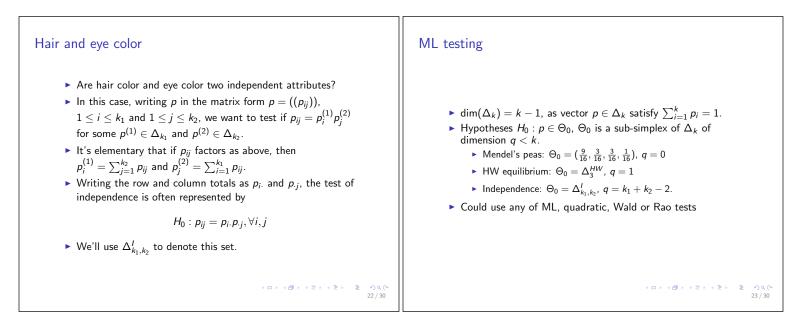
▶ The solution in *p* equals:

$$\hat{p}_{\text{MLE}} = \left(\frac{x_1}{n}, \cdots, \frac{x_k}{n}\right)$$

<ロ><一>、<一)>、<一)>、< き>、< き>、< き>、< き、< き、 16/30







Pearson's χ^2 tests

Rao's test statistics boils down to a very convenient form:

$$S(X) = \sum_{j=1}^{k} \frac{(X_j - n\hat{p}_{H_0,j})^2}{n\hat{p}_{H_0,j}}$$
$$= \sum \frac{(Observed - Expected)^2}{Expected}$$

- For categorical data, this was earlier discovered by Pearson who also ascertained its approximate χ^2 distribution.
- Because this is Rao statistic, we have $S(X) \xrightarrow{d} \chi^2(k-1-q)$



For testing H₀: p = p₀ against p ≠ p₀, where p₀ = (p_{0,1}, · · · , p_{0,k}) ∈ Δ_k is a fixed probability vector of interest (as in Mendel's peas example), then

$$S(X) = \sum_{j=1}^{k} \frac{(X_j - np_{0,j})^2}{np_{0,j}}$$

which is asymptotically $\chi^2(k-1-0) = \chi^2(k-1)$.

• Size- α Pearson's test rejects H_0 if $S(X) > q_{\chi^2}(1-\alpha, k-1)$.

<□><舂><≧><≧><≧>< 25/30

Example 2: parametric form Example 2: parametric form (contd.) • Once we have $\hat{\eta}_{\text{MLE}}$, we can construct ► HW test: $\hat{p}_{\mathcal{H}_0} = (\hat{\eta}_{\scriptscriptstyle\mathsf{MLE}}^2, \hat{\eta}_{\scriptscriptstyle\mathsf{MLE}}(1-\hat{\eta}_{\scriptscriptstyle\mathsf{MLE}}), \hat{\eta}_{\scriptscriptstyle\mathsf{MLE}}(1-\hat{\eta}_{\scriptscriptstyle\mathsf{MLE}}), (1-\hat{\eta}_{\scriptscriptstyle\mathsf{MLE}})^2) \text{ and }$ evaluate S(X). $H_0: p_0 = (\eta^2, \eta(1-\eta), \eta(1-\eta), (1-\eta)^2), \text{ for some } 0 < \eta < 1$ • Because Θ_0 has dimension q = 1 (only a single number η needs to be known), the asymptotic distribution of S(X) is ▶ To compute \hat{p}_{H_0} it is equivalent to write the likelihood $\chi^2(k-2).$ function in η and maximize: The same calculations carry through for a more general $L_X(\eta) = \text{const.} \times \{\eta^2\}^{x_1} \{\eta(1-\eta)\}^{x_2} \{\eta(1-\eta)\}^{x_3} \{(1-\eta)^2\}^{x_4}$ parametric form: $= \text{const.} imes \eta^{2x_1 + x_2 + x_3} (1 - \eta)^{x_2 + x_3 + 2x_4}$ $H_0: p = (g_1(\eta), \cdots, g_k(\eta))$ and so $\hat{\eta}_{\text{MLE}} = \frac{2x_1 + x_2 + x_3}{2n}$. where $\eta \in \mathcal{E}$ is *q*-dim vector and $g_1(\eta), \cdots, g_k(\eta)$ are functions such that for every $\eta \in \mathcal{E}$, $(g_1(\eta), \cdots, g_k(\eta)) \in \Delta_k$. <
 <

 <

Example 3: Independence
• Consider testing
$$H_0 : p_{ij} = p_i \cdot p_j, \forall i, j$$

• To get \hat{p}_{H_0} , we write the likelihood function in terms of $p_{i \cdot , 1 \leq i \leq k_1}$ and $p_{\cdot j}, 1 \leq j \leq k_2$:
 $L_x(p_1, \dots, p_{k_1}, p_{\cdot 1}, \dots, p_{\cdot k_2}) = \text{const.} \times \prod_{i=1}^{k_1} \prod_{j=1}^{k_2} (p_i \cdot p_j)^{x_{ij}}$
 $= \text{const.} \times \left\{ \prod_{i=1}^{k_1} p_{i}^{x_{i}} \right\} \left\{ \prod_{j=1}^{k_2} p_{j}^{x_j} \right\}$
where $x_{i \cdot} = \sum_{j=1}^{k_2} x_{ij}$ and $x_{\cdot j} = \sum_{i=1}^{k_1} x_{ij}$ are the margin counts
of our two-way table.

◆□ → ◆問 → ◆言 → ◆言 → 言 → りへで 28 / 30

. . . .

1.1

Example 3: independence (contd.)

▶ Because $(p_1, \dots, p_{k_1}) \in \Delta_{k_1}$ and $(p_{\cdot 1}, \dots, p_{\cdot k_2}) \in \Delta_{k_2}$, the maximizer is given by:

$$\hat{p}_{i.} = \frac{x_{i.}}{n}, \ \hat{p}_{.j} = \frac{x_{.j}}{n}, \ 1 \le i \le k_1, 1 \le j \le k_2$$

• And so \hat{p}_{H_0} has coordinates: $\hat{p}_{H_0,ij} = \frac{x_i \cdot x_j}{n^2}$ giving

$$S(X) = \sum_{i=1}^{k_1} \sum_{j=1}^{k_2} \frac{(X_{ij} - \frac{X_i, X_{ij}}{n})^2}{\frac{X_i, X_j}{n}}$$

► Because dim(Θ_0) = q = k₁ - 1 + k₂ - 1, $S(X) \xrightarrow{d} \chi^2(k_1k_2 - 1 - k_1 + 1 - k_2 + 1) = \chi^2((k_1 - 1)(k_2 - 1)).$

Hai	r-ey	ve color									
			Eye color								
			Blue	Green	Brown	Black	Total				
	r	Blonde	20 (13.9)	15 (13.6)	18 (28.4)	14 (11.0)	67				
	Hair color	Red	11(8.5)	4 (8.3)	24 (17.4)	2 (6.7)	41				
	air	Brown	9 (15.4)	11 (15.1)	36 (31.4)	18 (12.2)	74				
	Т	Black	8 (10.2)	17 (10.0)	20 (20.8)	4 (8.1)	49				
		Total	48	47	98	38	231				
• $S(x) = 30.9.$ • So the p-value is $1 - F_{(4-1)(4-1)}(30.9) = 1 - F_9(30.9) \approx 0.$											
					< □ >	< 18 > < 분 > < 분	া ছ 30/30				