**STA 215: Final Exam**

Time: 3 hours

**Name:** ________________________

| Qn | 1(a) | 1(b) | 1(c) | 1(d) | 2 | 3(a) | 3(b) | 4 | 5(a) | 5(b) | Total |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Points | | | | | | | | | | | |
| Max | 3 | 3 | 3 | 5 | 4 | 4 | 5 | 6 | 4 | 3 | 40 |

1. Let $X = (X_1, \cdots, X_n)$ with $X_i \overset{\text{IID}}{\sim} Poi(\mu)$ (pmf: $e^{-\mu}\mu^{x_i}/x_i!$ for $x_i = 0, 1, \cdots$) with $\mu > 0$ unknown.

   (a) Show that the Fisher information is given by $I^F(\mu) = n/\mu$. [3 points]

   (b) Is $\hat{\mu}_{\text{MLE}} = \bar{X}$ a uniformly minimum variance unbiased estimator (UMVUE) of $\mu$? Explain. [3 points]

   (c) Identify the posterior distribution of $\mu$ under the Jeffreys prior $\pi_J(\mu) \propto \mu^{-1/2}$ (give name of the distribution and exact expressions for parameters). [3 points]

   (d) Derive the Wald and Rao test statistics for testing $H_0 : \mu = \mu_0$ vs. $\mu \neq \mu_0$. For this problem, is one practically more useful than the other? Explain. [5 points]

2. Consider $X = (X_1, \cdots, X_n)$ with $X_i$'s independently distributed as $X_i \sim N(\mu_i, 1)$. For testing $H_0 : \mu_1 = \mu_2 = \cdots = \mu_n = 0$ vs. $H_1$ : at least one $\mu_i \neq 0$, let $\delta$ be a test that rejects $H_0$ if any $|X_i|$ exceeds $q_N(1 - \alpha/2)$. Is $\delta$ a level-$\alpha$ test? Justify your answer. [4 points]

3. Consider a statistical model $X \sim p(x|\theta)$, $\theta \in \Theta$ and suppose we want to test $H_0 : \theta \in \Theta_0$ vs. $\theta \in \Theta_1$ where $\Theta_0$ and $\Theta_1$ form a partition of $\Theta$. Let $T = T(X)$ be a statistic such that large values of $|T|$ provide evidence against $H_0$.

   (a) Suppose $S = S(X)$ is another statistic such that the distribution of $T$ given $S$ is the same for all $\theta \in \Theta_0$. Denote the common cumulative distribution function of $T$ given $S = s$, under every $\theta \in \Theta_0$, by $F(t|s)$, $-\infty < t < \infty$. As usual, let $q_F(u, s)$ give the quantiles of $F(t|s)$ for $u \in (0, 1)$. Show that for any $\alpha \in (0, 1)$, the test that rejects $H_0$ for $T > q_F(1 - \alpha/2, S)$ or $T < q_F(\alpha/2, S)$ is level-$\alpha$. [4 points]

(b) Consider $X = (X_1, X_2)$ where $X_1$ and $X_2$ are modeled as independent exponential random variables with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$ (i.e., for $i = 1, 2$, the pdf of $X_i$ is $\lambda_i \exp(-\lambda_i x_i)$, $x_i > 0$). A simple calculation shows that the joint pdf of $S = X_1 + X_2$ and $T = X_1 - X_2$ is given by,

$$f(t, s|\lambda_1, \lambda_2) = \frac{\lambda_1 \lambda_2}{2} \exp\left\{-\frac{(\lambda_1 + \lambda_2)s + (\lambda_1 - \lambda_2)t}{2}\right\}, \quad s > 0, -s < t < s,$$

and $f(t, s|\lambda_1, \lambda_2) = 0$ otherwise. Argue that for testing $H_0 : \lambda_1 = \lambda_2$ against $H_1 : \lambda_1 \neq \lambda_2$, a level-$\alpha$ test is given by "Reject $H_0$ if $|T| > (1 - \alpha)S$". [5 points]

4. Consider again $X = (X_1, X_2)$ where $X_1$ and $X_2$ are modeled as independent exponential random variables with parameters $\lambda_1 > 0$ and $\lambda_2 > 0$. Describe the level-$\alpha$ maximum likelihood test (a.k.a., the likelihood ratio test) for testing $H_0 : \lambda_1 = \lambda_2$ against $H_1 : \lambda_1 \neq \lambda_2$ (write down your test as "reject $H_0$ if $T > c$", give a simple expression for $T$ and identify $c$ in terms of $\alpha$ and possibly a quantile of a named distribution with exact expressions for its parameters). [Hint: $u(1 - u)$ is small if $|u - 1/2|$ is large.] [6 points]

5. A researcher has recruited $n$ subjects for a study with two treatment conditions C1 and C2. She models the outcome as $Y_i = z_i^T \beta + \epsilon_i$, $\epsilon_i \overset{\text{IID}}{\sim} N(0, \sigma^2)$ with $z_i = (A_i, C_{1i}, C_{2i})$ where $A_i$ is the age of subject $i$, $C_{1i} = 1, C_{2i} = 0$ if subject $i$ is assigned to C1, $C_{1i} = 0, C_{2i} = 1$ otherwise. Letting $Z$ denote the design matrix, it can be shown that

$$Z^T Z = \begin{pmatrix} \sum_i A_i^2 & n_1 \bar{A}_1 & n_2 \bar{A}_2 \\ n_1 \bar{A}_1 & n_1 & 0 \\ n_2 \bar{A}_2 & 0 & n_2 \end{pmatrix},$$

$$(Z^T Z)^{-1} = \frac{1}{n_1 n_2 (n_1 s_1^2 + n_2 s_2^2)} \begin{pmatrix} n_1 n_2 & -n_1 n_2 \bar{A}_1 & -n_1 n_2 \bar{A}_2 \\ -n_1 n_2 \bar{A}_1 & n_2 \sum_i A_i^2 - n_2^2 \bar{A}_2^2 & n_1 n_2 \bar{A}_1 \bar{A}_2 \\ -n_1 n_2 \bar{A}_2 & n_1 n_2 \bar{A}_1 \bar{A}_2 & n_1 \sum_i A_i^2 - n_1^2 \bar{A}_1^2 \end{pmatrix}$$

where $n_i$, $\bar{A}_i$ and $s_i^2$ denote the number, average age and variance of age of subjects assigned to C$i$, $i = 1, 2$. For this problem, variance is defined as follows: the variance of $x_1, \cdots, x_k$ is $\frac{1}{k} \sum_{i=1}^k (x_i - \bar{x})^2 = \frac{1}{k} \sum_i x_i^2 - \bar{x}^2$.

(a) Assume $\sigma^2$ is known and consider the flat prior $\pi(\beta) \propto 1$ (which leads to the posterior $\pi(\beta|y) = N(\hat{\beta}_{\text{LS}}, \sigma^2 (Z^T Z)^{-1})$). What is the posterior variance of the treatment contrast $\beta_3 - \beta_2$? Simplify. [4 points]

(b) Suppose $n = 2m$ and the researcher decides to assign $m$ subjects to each group. Further suppose she has the age records available to her before she makes this assignment. Can she do any better job than just randomly splitting them into two equal halves? Explain (I don't need a mathematical proof. A careful reasoning would do!). [3 points]