Problem 1a.) Following the hint, we solve for λ and get

$$\begin{aligned}
0 &= -2(z'z)(\beta - \hat{\beta}) + a\lambda \\
0 &= -2a'(z'z)^{-1}(z'z)(\beta - \hat{\beta}) + a'(z'z)^{-1}a\lambda \\
\lambda &= 2(a'(z'z)^{-1}a)^{-1}(a'\beta - a'\hat{\beta}) \\
\tilde{\lambda} &= 2(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta}) && \text{using parital derivative wrt } \lambda.
\end{aligned}$$

Now plug this in and solve for β.

$$\begin{aligned}
0 &= -2(z'z)(\beta - \hat{\beta}) + 2a(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta}) \\
(\beta - \hat{\beta}) &= (z'z)^{-1}a(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta}) \\
\tilde{\beta} &= (z'z)^{-1}a(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta}) + \hat{\beta}.
\end{aligned}$$

Because of how we set up the langrage function, $\tilde{\beta}$ gives us the value of β that maximizes $g(\beta)$, subject to the constraint that $a'\beta = \eta$. If we plug $\tilde{\beta}$ into $g(\beta)$, we will get $h(\eta)$. So,

$$\begin{aligned}
h(\eta) &= g(\tilde{\beta}) \\
&= \left((z'z)^{-1}a(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta})\right)^T (z'z) \left((z'z)^{-1}a(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta})\right) \\
&= \left((\eta - \hat{\eta})^T(a'(z'z)^{-1}a)^{-1}a'(z'z)^{-1}\right) (z'z) \left((z'z)^{-1}a(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta})\right) \\
&= (\eta - \hat{\eta})^T(a'(z'z)^{-1}a)^{-1}(a'(z'z)^{-1}a)(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta}) \\
&= (\eta - \hat{\eta})^T(a'(z'z)^{-1}a)^{-1}(\eta - \hat{\eta})
\end{aligned}$$

as desired.

Problem 1b) $l^*(\eta)$ is defined as the maximum value of $-f(-g(\beta))$ subject to the constraint that $a'\beta = \eta$. Part a. proves that $h(\eta)$ is the value of $g(\beta)$ maximized under that constraint. So

$$\begin{aligned}
h(\eta) &\text{ is } max(g(\beta)) \text{ given } a'\beta = \eta, \\
\Rightarrow -h(\eta) &\text{ is } min(-g(\beta)) \text{ given } a'\beta = \eta, \\
\Rightarrow f(-h(\eta)) &\text{ is } min(f(-g(\beta))) \text{ given } a'\beta = \eta \text{ because f is monotonic,} \\
\Rightarrow -f(-h(\eta)) &\text{ is } max(-f(-g(\beta))) \text{ given } a'\beta = \eta,
\end{aligned}$$

as desired.

Problem 2.) We are given in the question that for the hypothesis test $H_0 : a'\beta \leq \eta_0$, we can define our rejection region as $\{\hat{\beta} : \eta_0 \leq a'\hat{\beta} - cs/\sqrt{n_a}$ for some fixed cutoff c. We must show the size is $1 - F_{n-p}(c)$, where $F_\nu(x)$ is the cdf function of a t-distribution with ν df evaluated at x. The size of a test is defined as,

$$\begin{aligned}
p(\text{Reject } H_0|H_0) &= p(\eta_0 \leq a'\hat{\beta} - cs/\sqrt{n_a}|H_0) \\
&= p(\eta_0 - a'\beta \leq a'\hat{\beta} - a'\beta - cs/\sqrt{n_a}|H_0) \\
&= p(\frac{\eta_0 - a'\beta}{s/\sqrt{n_a}} + c \leq \frac{a'\hat{\beta} - a'\beta}{s/\sqrt{n_a}}|H_0).
\end{aligned}$$

In the notes, it was shown that $\frac{a'\hat{\beta}-a'\beta}{s/\sqrt{n_a}} \sim t_{n-p}$. Let $T \sim t_{n-p}$, then

$$= p(\frac{\eta_0 - a'\beta}{s/\sqrt{n_a}} + c \leq T|H_0)$$

Note that $\frac{\eta_0-a'\beta}{s/\sqrt{n_a}}$ is a decreasing function in $a'\beta$, making $p(\frac{\eta_0-a'\beta}{s/\sqrt{n_a}} + c \leq T)$ increasing in $a'\beta$. Since H_0 is assumed, the largest $a'\beta$ can be is η_0. Therefore,

$$\begin{aligned}
&\leq p(\frac{\eta_0 - \eta_0}{s/\sqrt{n_a}} + c \leq T) \\
&= p(c \leq T) \\
&= 1 - F_{n-p}(c)
\end{aligned}$$

as desired.

Problem 3a)

- $z'z = \begin{bmatrix} n_1 & 0 \\ 0 & n_2 \end{bmatrix}$
- $\hat{\beta} = (\hat{u}, \hat{v})^T$
- $s^2 = \frac{(n_1-1)s_u^2+(n_2-1)s_v^2}{n_1+n_2-2}$

Problem 3b) Using the fact given in the notes that $\frac{a'\hat{\beta}-a'\beta}{s/\sqrt{n_a}} \sim t_{n-p}$, we can see that

$$p(t_{n-p,\frac{\alpha}{2}} \leq \frac{a'\hat{\beta} - a'\beta}{s/\sqrt{n_a}} \leq t_{n-p,1-\frac{\alpha}{2}}) = 1 - \alpha$$

where $t_{n-p,\alpha}$ is the inverse CDF function for a t_{n-p} evaluated at α. Let $t^* = t_{n-p,1-\frac{\alpha}{2}}$. Doing some algebraic manipulations, we get

$$p(a'\hat{\beta} - t^*\frac{s}{\sqrt{n_a}} \leq a'\beta \leq a'\hat{\beta} + t^*\frac{s}{\sqrt{n_a}}) = 1 - \alpha.$$

If we let $a' = (1, -1)$ and remember that $n_a = 1/a'(z'z)^{-1}a$, then we get

$$p(\bar{u}-\bar{v}-t^*\sqrt{(\frac{(n_1-1)s_u^2+(n_2-1)s_v^2}{n_1+n_2-2})(\frac{1}{n_1}+\frac{1}{n_2})}$$

$$\leq \mu_1-\mu_2 \leq \bar{u}-\bar{v}+t^*\sqrt{(\frac{(n_1-1)s_u^2+(n_2-1)s_v^2}{n_1+n_2-2})(\frac{1}{n_1}+\frac{1}{n_2})}) = 1-\alpha.$$

Which means by definition that the interval $\bar{u}-\bar{v}\pm t^*\sqrt{(\frac{(n_1-1)s_u^2+(n_2-1)s_v^2}{n_1+n_2-2})(\frac{1}{n_1}+\frac{1}{n_2})}$ gives a 95% CI for $\mu_1-\mu_2$.

Problem 3c) We can easily construct a size α test for $H_0 : \mu_1-\mu_2 = 0$ by rejecting H_0 if the confidence interval in part b does not contain 0. The p-value will be the size α that gives us 0 being exactly on the boundary of the confidence interval. This is because a size any larger will reject and a size as small or any smaller will fail to reject. From our data, we get the $(1-\alpha)$CI to be $(19 \pm t^* * 10.05)$. We need to find the t^* st:

$$\begin{aligned}
0 &= 19 - t^* * 10.05 \\
t^* &= 1.89 \\
1-\frac{\alpha}{2} &= F_{17}(1.89) \\
1-\frac{\alpha}{2} &= 0.962 \\
\alpha &= 0.076
\end{aligned}$$

which means that our p-value is 0.076.

Problem 3d) Using the test from question 2., we need to find the size α that puts $\eta_0 = 0$ on the boundary of the rejection region. To do this, we need c st.

$$\begin{aligned}
0 &= a'\hat{\beta} - c * s/\sqrt{n_a} \\
c &= \frac{a'\hat{\beta}}{s/\sqrt{n_a}} \\
c &= \frac{19}{10.05} \\
c &= 1.89.
\end{aligned}$$

Since we know the size of the test from 2.) as a function of c, we can find the size that gives that $c = 1.96$:

$$\begin{aligned}
\alpha &= 1 - F_{n-p}(1.89) \\
\alpha &= 1 - 0.962 \\
\alpha &= 0.038
\end{aligned}$$

so our p=value is 0.038. Note, since our calculation in the two tailed boiled down to $1 - F(c) = \frac{\alpha}{2}$ and for the one tailed it was $1 - F(c) = \alpha$, the one tailed p-value will always be 1/2 of the two tailed p-value.

Problem 4a) From the notes, we know that $\hat{\beta}$ is approximately $N(\beta, I_x^{-1})$, which means $\frac{\hat{\beta}-\beta}{\sqrt{I_x^{-1}}} \sim N(0,1)$. Using the same logic from 3b), a 95% CI can be found using $\hat{\beta} \pm 1.96 * \sqrt{I_x^{-1}}$. Using numbers given, this is $1.006264 \pm 1.96 * \sqrt{(}1/752132.7)$ which gives the interval $(1.004, 1.00852)$.

Problem 4b) Using the same logic as 3c) (except the pivotal quantity is normal instead of t), the p-value is:

$$
\begin{aligned}
p &= 2 * (1 - \Phi(\frac{\hat{\beta} - 1}{\sqrt{I_x^{-1}}})) \\
&= 2 * (1 - \Phi(5.43)) = 5.6 * 10^{-8}
\end{aligned}
$$

In identically parallel logic to question 3d), the p-value in the one-sided case here will be 1/2 the 2-sided case. This means the one-sided p-value is $2.8 * 10^{-8}$.

Problem 4c) From the notes, we have $a'\hat{\theta}$ is approximately $N(a'\theta, a'I_x^{-1}a)$. Since $\theta = (\alpha, \beta)^T$, if we use $a' = (0, 1)$, then we get $\hat{\beta}$ is approximately $N(\beta, a'I_x^{-1}a)$. Using this, I can make the pivotal quantity

$$
\frac{1.006264 - 1}{0.001151243} = 5.441077
$$

Note in R, if you are trying to get the inverse of a matrix, you cannot use X^{-1}, which gives element-wise inverses. In simple cases like this, you can use the command SOLVE.

The p-value is then $2(1-\Phi(5.44)) = 5.3 * 10^{-8}$, which is very close to the profile likelihood approximation in part b. The p-value for the one-sided test is $2.65 * 10^{-8}$, again very close to the approximation in part b.

Problem 5a) The coverage of a confidence interval is defined as $p(\theta_0 \in CI | \theta = \theta_0)$. Recall from the last homework we found

- $A_{1/2}(x = 0) = \{0\}$
- $A_{1/2}(x = 1) = \{0, 1\}$
- $A_{1/2}(x = 2) = \{1, 2\}$
- $A_{1/2}(x = 3) = \{3\}$

So the idea is for each different θ_0, we need to find the values of x that put θ_0 in $A_{1/2}$, and then find the probability that x is any one of those values given $\theta = \theta_0$

- $\gamma(\theta = 0, A_{1/2}) = P(0 \in A_{1/2} | \theta = 0) = P(x = 0 | \theta = 0) + P(x = 1 | \theta = 0) = 7/10$
- $\gamma(\theta = 1, A_{1/2}) = P(1 \in A_{1/2} | \theta = 1) = P(x = 1 | \theta = 1) + P(x = 2 | \theta = 1) = 5/6$
- $\gamma(\theta = 2, A_{1/2}) = P(2 \in A_{1/2} | \theta = 2) = P(x = 2 | \theta = 2) = 2/3$

- $\gamma(\theta = 3, A_{1/2}) = P(3 \in A_{1/2}|\theta = 3) = P(x = 3|\theta = 3) = 1$

Problem 5b) The confidence coefficient is $inf_\theta \gamma(\theta, A_{1/2}) = 2/3$.

Problem 6a.) To do this, we need to find the coverage of A_k for any specific θ_0. Using what we found in HW1, we have

$$\begin{aligned} P(\theta_0 \in A_k|\theta = \theta_0) &= P(\frac{1}{\theta_0^n} 1_{\theta_0 > max(x_i)} \geq k \frac{1}{max(x_i)^n}|\theta = \theta_0) \\ &= P((\frac{1}{k})^{1/n} max(x_i) \geq \theta_0 \geq max(x_i)|\theta = \theta_0) \end{aligned}$$

We know $P(\theta_0 \geq max(x_i)|\theta = \theta_0) = 1$, so this condition can be dropped from the probability statement.

$$\begin{aligned} &= P((\frac{1}{k})^{1/n} max(x_i) \geq \theta_0|\theta = \theta_0) \\ &= P(max(x_i) \geq k^{1/n}\theta_0|\theta = \theta_0) \\ &= 1 - P(max(x_i) \leq k^{1/n}\theta_0|\theta = \theta_0) \\ &= 1 - P(x_1, ..., x_n \leq k^{1/n}\theta_0|\theta = \theta_0) \\ &= 1 - \prod P(x_i \leq k^{1/n}\theta_0|\theta = \theta_0) && \text{since iid} \\ &= 1 - P(x_1 \leq k^{1/n}\theta_0|\theta = \theta_0)^n && \text{since iid} \\ &= 1 - (\frac{k^{1/n}\theta_0}{\theta_0})^n \\ &= 1 - k. \end{aligned}$$

So the coverage of the set does not depend on the value of θ, which means the confidence coefficient is $1 - k$.

Problem 6b) Using part a., we know $A_{0.05}$ will have coverage $1 - 0.05 = 0.95$. $A_{0.05}$ for this data was found in HW1, and was $(max(x_i), (1/k)^{1/n} max(x_i)) = (19.6, 26.45)$.

Problem 6c) No matter what α is, the left side of the interval will always be at 19.6. So we must find the value of α that puts $\theta = 30$ on the right boundary. Note, using part a. we can say $\alpha = k$. Thus, we need

$$
\begin{aligned}
30 &= (1/k)^{1/n} max(x_i) \\
30 &= (1/k)^{(1/10)} 19.6 \\
k &= (\frac{19.6}{30})^{10} \\
k &= 0.014.
\end{aligned}
$$

Problem 6d) Yes, this is the p-value for $H_0 = 30$ for this data. Note that since we are constructing a test based on a confidence interval, the rejection region is defined as the set of values of the test statistic for which $\theta = 30$ is not in A_α. The p-value is defined as the size p for which if $\alpha > p$, we would reject, but if $\alpha \leq p$, we would fail to reject. In this case, decreasing α only makes the right boundary of A_α larger. Since $\alpha = 0.014$ puts $\theta = 30$ directly on the right boundary, if $\alpha > 0.014$, then the right boundary for A_α would be less than 30, which means that $\theta = 30$ would not be in A_α and we would reject $H_0 : \theta = 30$. But if $\alpha \leq 0.014$, then the right boundary of A_α would be larger than 30, meaning $30 \in A_\alpha$ and would would fail to reject H_0.