Lévy Random Fields

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1 Poisson Random Measures

Let $\big(\mathcal{X}, \mathcal{F}, \nu(dx)\big)$ be a σ-finite measure space. One way to construct a random measure $H(dx) \sim \mathsf{Po}(\nu)$ on $(\mathcal{X}, \mathcal{F})$ with mean ν, so $H(A) \sim \mathsf{Po}\big(\nu(A)\big)$ for each $A \in \mathcal{F}$, is:

- Partition $\mathcal{X} = \cup \Lambda_j$ into $J \le \infty$ disjoint sets $\Lambda_j \in \mathcal{F}$ of positive finite measure $\lambda_j := \nu(\Lambda_j) \in (0, \infty)$, for $0 \le j < J$;
- Draw independent Poisson random variables $n_j \sim \mathsf{Po}(\lambda_j)$;
- Set $\nu_j(A) := \nu(A \cap \Lambda_j)/\lambda_j$ for $A \in \mathcal{F}$, the probability measure arising from restricting ν to Λ_j and normalizing. Then draw independent random samples $\{x_{ij} : \ 0 \le i < n_j\} \overset{\text{iid}}{\sim} \nu_j(dx)$ on each Λ_j;
- Set $H(dx) := \sum_{i,j} \delta_{x_{ij}}(dx)$ (the sum of a unit point mass at each x_{ij}) or, equivalently, for $A \in \mathcal{F}$ set $H(A) = \sum_{j<J} \#\left[A \cap \{x_{ij} : \ i < n_j\}\right]$.

Another (more abstract) approach is simply to note that the assignment of probability distributions $A \mapsto \mathsf{Po}\big(\nu(A)\big)$ to elements $A \in \mathcal{F}$ of finite measure $\nu(A) < \infty$ is consistent in Kolmogorov's sense, so there exists some probability space $(\Omega, \mathcal{F}, \mathsf{P})$ and an assignment $A \mapsto H(A) \sim \mathsf{Po}\big(\nu(A)\big)$ of random variables on that space with the specified distributions.

Let's begin with such a random Poisson measure $H \sim \mathsf{Po}\big(\nu(dx)\big)$ on $(\mathcal{X}, \mathcal{F}, \nu)$. Of course the characteristic function (ch.f.) of $H(A)$ is

$$\mathsf{E}\left[e^{i\omega H[A]}\right] = \sum_{k=0}^{\infty} e^{i\omega k} \left\{ \frac{\nu(A)^k}{k!} \ e^{-\nu(A)} \right\} = \exp\left\{ \big(e^{i\omega} - 1\big) \nu(A) \right\}.$$

1.1 Musielak-Orlicz spaces I

By linearity, the "stochastic integral" of a simple function

$$f = \sum a_n \mathbf{1}_{\{A_n\}}$$

(finite sum, with $A_i \cap A_n = \emptyset$ for $i \neq j$) has to be

$$H[f] := \int_{\mathcal{X}} f(x)\, H(dx) = \sum a_n H(A_n) = \sum f(x_j)$$

where $\{x_j\} = \operatorname{spt}(H) \subset \mathcal{X}$ is the (random and countable) support of H. By independence of $\{H(A_n)\}$ the ch.f. will be a product, with log ch.f.

$$\begin{aligned} \log \mathsf{E}\big[e^{i\omega H[f]}\big] &= \sum_n \big[e^{i\omega a_n} - 1\big]\, \nu(A_n) \\ &= \int_{\mathcal{X}} \Big[e^{i\omega f(x)} - 1\Big]\, \nu(dx). \end{aligned} \tag{1}$$

Of course we can extend $H[f]$ to limits $f \in L_1(\mathcal{X}, \mathcal{F}, \nu)$ of simple functions by L_1-continuity, but we can go further. We can extend the definition of $H[f]$, and continue to satisfy Eqn (1), to any f for which the integral in (1) is well-defined— *i.e.*, to f in the space

$$\Psi_{0\wedge 1} := \left\{ f : \int_{\mathcal{X}} \big(1 \wedge |f(x)|\big)\, \nu(dx) < \infty \right\}. \tag{2}$$

For more details on "Musielak-Orlicz" (M-O) spaces like this, see (Rajput and Rosiński 1989, Thm. 3.3) and (Kallenberg 2002); also (Gaigalas 2004a,b). The mean and variance, when they exist, are given by

$$\mathsf{E}H[f] = \int_{\mathcal{X}} x\nu(dx) \qquad \mathsf{V}H[f] = \int_{\mathcal{X}} x^2\nu(dx).$$

1.2 Infinitely-Divisible Distributions

A distribution μ on $\mathbb{R}^d$ is called *infinitely divisible* or simply *ID* if, for each $n \in \mathbb{N}$, $\mu = (\mu_n)^{*n}$ is the n-fold convolution of some other distribution μ_n— or, equivalently, if any random variable $X \sim \mu(dx)$ may be written for each $n \in \mathbb{N}$ as the sum

$$X = X_1 + \cdots + X_n$$

of n iid random variables X_j. Familiar examples include the Normal, Poisson, Gamma, Negative Binomial, Inverse Gaussian, and α-Stable distributions. Khinchine and Lévy (1936) showed that each such distribution has a log characteristic function of the form

$$\begin{aligned}\log \mathsf{E} e^{i\omega' X} &= \log \int_{\mathbb{R}^d} e^{i\omega' u} \mu(du) \\ &= i\omega' m - \tfrac{1}{2}\omega' \Sigma \omega + \int_{\mathbb{R}^d} \big(e^{i\omega' u} - 1\big) \nu(du) \end{aligned} \tag{3a}$$

for some $m \in \mathbb{R}^d$, positive definite $d \times d$ matrix Σ, and σ-finite Borel measure $\nu(du)$ (called the "Lévy measure") satisfying

$$\int_{\mathbb{R}^d} \big(1 \wedge |u|\big)\, \nu(du) < \infty \tag{3b}$$

or, more generally (ignore this part until Section (3)),

$$\log \mathsf{E} e^{i\omega X} = i\omega' m - \tfrac{1}{2}\omega' \Sigma \omega + \int_{\mathbb{R}^d} \big[e^{i\omega' u} - 1 - i\omega' h(u)\big] \nu(du) \tag{4a}$$

with σ-finite Lévy measure $\nu(du)$ satisfying the weaker condition

$$\int_{\mathbb{R}^d} \big(1 \wedge u^2\big)\, \nu(du) < \infty \tag{4b}$$

and bounded function $h(u) = u + O(|u|^2)$. We'll return to Eqn (4) below; let's consider Eqn (3) first.

The case $\nu \equiv 0$ is simply the multivariate normal $X \sim \mathsf{No}(m, \Sigma)$; by Eqn (3) or Eqn (4), any ID random variable may be written as the sum of an independent normal random variable with the $\mathsf{No}(m, \Sigma)$ distribution, and an ID random variable with both m and Σ zero. To simplify formulas we'll take $m = 0$ and $\Sigma = 0$ in the sequel and omit them. Denote by $\mathsf{ID}(\nu)$ the distribution μ with Lévy measure ν, so $\mu \sim \mathsf{ID}(\nu)$ means that $(\forall \omega \in \mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} e^{i\omega' x}\, \mu(dx) = \exp\left\{ \int_{\mathbb{R}^d} \big(e^{i\omega' u} - 1\big)\, \nu(du) \right\}. \tag{5}$$

1.2.1 Examples

The **Poisson** distribution $\mathsf{Po}(\mu)$ with mean μ is ID, since any $X \sim \mathsf{Po}(\mu)$ may be written as the sum of n random variables $X_j \overset{\text{iid}}{\sim} \mathsf{Po}(\mu/n)$; its ch.f.

$$\mathsf{E} e^{i\omega X} = \sum_{k=0}^{\infty} e^{i\omega k} \left\{ \mu^k\, e^{-\mu}/k! \right\} = \exp\left[(e^{i\omega} - 1)\, \mu \right]$$

is of the form Eqn (3a) for Lévy measure $\nu(du) = \mu\,\delta_1(du)$, a point mass of magnitude $\mu > 0$ at $u = 1$. This clearly satisfies Eqn (3b) with $d = 1$, so $\mathsf{Po}(\mu) = \mathsf{ID}(\mu\delta_1)$.

The **Gamma** distribution $\mathsf{Ga}(\alpha,\beta)$ with shape parameter α and rate parameter β has ch.f.

$$\mathsf{E}e^{i\omega X} = \int_0^\infty e^{i\omega x}\left\{\beta^\alpha\, x^{\alpha-1}e^{-\beta x}/\Gamma(\alpha)\right\}\,dx = (1 - i\omega/\beta)^{-\alpha}$$

which will be of form Eqn (3a) if some measure ν satisfies

$$\int_{\mathbb{R}} \big(e^{i\omega u} - 1\big)\nu(du) \overset{?}{=} \alpha\log\beta - \alpha\log(\beta - i\omega).$$

If $\nu(du)$ has density $\nu(u)$, upon differentiating wrt ω, the requirement becomes

$$\int_{\mathbb{R}} e^{i\omega u}\,\{iu\nu(u)\}\;du = \frac{i\alpha}{\beta - i\omega}.$$

By Fourier inversion,

$$u\nu(u) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{-i\omega u}\,\frac{\alpha}{\beta - i\omega}\,d\omega = \alpha\,e^{-\beta u}\mathbf{1}_{\{u>0\}},$$

so $\nu(du) = \alpha\,u^{-1}\,e^{-\beta u}\,du$ on $\mathbb{R}_+$. This again satisfies Eqn (3b), so $\mathsf{Ga}(\alpha,\beta) = \mathsf{ID}\big(\alpha u^{-1}e^{-\beta u}\mathbf{1}_{\{u>0\}}du\big)$.

The **Negative Binomial** distribution $\mathsf{NB}(\alpha, p)$ with shape parameter α and success probability $p = (1-q) \in (0,1)$ has ch.f.

$$\begin{aligned}\mathsf{E}e^{i\omega X} &= \sum_{k=0}^\infty e^{i\omega k}\left\{\binom{k+\alpha}{k}p^\alpha q^k\right\}\\ &= \sum_{k=0}^\infty e^{i\omega k}\left\{\binom{-\alpha}{k}p^\alpha(-q)^k\right\} = p^\alpha\big(1 - q\,e^{i\omega}\big)^{-\alpha}\end{aligned}$$

which will be of form Eqn (3a) if for some ν

$$\int_{\mathbb{R}} \big(e^{i\omega u} - 1\big)\nu(du) = \alpha\log p - \alpha\log(1 - q\,e^{i\omega}).$$

If $\nu(du)$ has a point mass of size ν_k at each integer $k \in \mathbb{N}$, write $z = e^{i\omega}$ and differentiate wrt z to get:

$$\sum_{k=1}^\infty z^{k-1}\,\{k\nu_k\} = \frac{\alpha\,q}{1 - qz} = \alpha\,q\sum_{j=0}^\infty (qz)^j.$$

Upon matching powers of z (for $k = j+1$), we have $\nu_k = \alpha q^k/k$ so $\nu(du) = \sum_{k\in\mathbb{N}} \frac{\alpha q^k}{k}\,\delta_k(du)$, which obviously satisfies Eqn (3b).

The **Symmetric α-Stable** (or SαS) distribution $\mathsf{St}_\mathsf{A}(\alpha, 0, \gamma, 0)$ has ch.f.

$$\mathsf{E}e^{i\omega X} = e^{-\gamma|\omega|^\alpha}$$

for some $0 < \alpha \le 2$ and $\gamma > 0$; the best-known examples are the Cauchy (with $\alpha = 1$) and Normal (with $\alpha = 2$ and $\sigma^2 = 2\gamma$). The SαS would be of form Eqn (3a) with absolutely continuous Lévy measure $\nu(du) = \nu(u)\,du$ if

$$\int_{\mathbb{R}} \big(e^{i\omega u} - 1\big)\nu(u)\,du = -\,\gamma|\omega|^\alpha. \tag{6}$$

This is an even function of ω, so $\nu(u) = \nu(-u)$ for all $u \in \mathbb{R}$ and

$$-\gamma|\omega|^\alpha = \int_{\mathbb{R}} \big[\cos(\omega u) - 1\big]\nu(u)\,du. \tag{7}$$

Setting $\omega = 1$ and then changing variables $u \rightsquigarrow |\omega|u$, this gives

$$-\gamma = \int_{\mathbb{R}} \big[\cos(u) - 1\big]\nu(u)\,du = \int_{\mathbb{R}} \big[\cos(\omega u) - 1\big]\nu(\omega u)\,|\omega|\,du$$

Multiplying by $|\omega|^\alpha$ and comparing to Eqn (7),

$$\int_{\mathbb{R}} \big[\cos(\omega u) - 1\big]\nu(u)\,du = \int_{\mathbb{R}} \big[\cos(\omega u) - 1\big]\nu(\omega u)\,|\omega|^{\alpha+1}\,du$$

for all ω and hence $\nu(u) \equiv \nu(\omega u)\,|\omega|^{\alpha+1}$ or, setting $\omega = 1/u$,

$$\nu(u) = \nu(1)|u|^{-\alpha-1}$$

for all $u \in \mathbb{R}$. We can evaluate $\nu(1)$ by taking the limit as $\epsilon \to 0$ of Eqn (6) with $\omega = 1 + i\epsilon$. First multiply by α, and integrate by parts:

$$\begin{aligned}
-\gamma\alpha &= \lim_{\epsilon\searrow 0} \Re\left\{\int_{\mathbb{R}} [e^{-(\epsilon-i)u} - 1]\,\alpha|u|^{-\alpha-1}\,du\right\}\nu(1)\\
&= \nu(1)\lim_{\epsilon\searrow 0}\Re\{(\epsilon-i)^\alpha\}\,2\int_0^\infty [e^{-x} - 1]\,\alpha x^{-\alpha-1}\,dx\\
&= \nu(1)\cos\left(\tfrac{\pi\alpha}{2}\right)\left\{-2\int_0^\infty \left(-e^{-x}\right)\left(-x^{-\alpha}\right)\right\}dx\\
&= \nu(1)\cos\left(\tfrac{\pi\alpha}{2}\right)\{-2\Gamma(1-\alpha)\}\,.
\end{aligned}$$

Using the identities $z\Gamma(z)\Gamma(-z)\sin(\pi z) \equiv -\pi$ and $\sin(\pi\alpha) = 2\sin(\frac{\pi\alpha}{2})\cos(\frac{\pi\alpha}{2})$, we can compute $\nu(1) = \frac{\gamma}{\pi}\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}$ so finally $\mathsf{St_A}(\alpha, 0, \gamma, 0) = \mathsf{ID}(\nu)$ for

$$\nu(du) = \frac{\gamma}{\pi}\Gamma(\alpha)\sin(\tfrac{\pi\alpha}{2})\alpha|u|^{-\alpha-1}\,du = \gamma c_\alpha \alpha |u|^{-\alpha-1}\,du \tag{8}$$

where we introduce $c_\alpha = \frac{1}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}$ to simplify formulas. For this to satisfy Eqn (3b) we would need finiteness for

$$\begin{aligned}\int_{\mathbb{R}} \big(1 \wedge |u|\big)\,\nu(du) =& 2c_\alpha \int_0^1 \alpha u^{-\alpha}\,du + 2c_\alpha \int_1^\infty \alpha u^{-\alpha-1}\,du \\ =& 2c_\alpha/(1-\alpha) \text{ if } 0 < \alpha < 1, \text{ otherwise } \infty,\end{aligned}$$

i.e., we would need $0 < \alpha < 1$— and, in particular, the Cauchy distribution with $\alpha = 1$ and $\nu(du) = (\gamma/\pi)u^{-2}\,du$ is not quite included.

1.3 Aside on the Connection between ν and μ

Any ID distribution $\mu(dx) = \mathsf{ID}(\nu(du))$ with Lévy measure $\nu(du)$, may be viewed as $\mu_1(dx)$ in an additive convolution semigroup $\{\mu_\theta(dx)\}_{\theta \geq 0}$ of distributions $\mu_\theta \sim \mathsf{ID}(\theta\,\nu)$ with Lévy measure $\theta\nu(du)$. For $X \sim \bar{\mu}(dx)$ and large $n \in \mathbb{N}$, we have

$$\begin{aligned}\mathsf{E}\left[e^{i\omega X}\right] &= \int e^{i\omega x}\,\mu_1(dx) \\ &= \left[\int e^{i\omega x}\,\mu_{1/n}(dx)\right]^n \\ &= \left[1 + \int \left\{e^{i\omega x} - 1\right\}\,\mu_{1/n}(dx)\right]^n \\ &= \left[1 + \frac{1}{n}\int \left\{e^{i\omega x} - 1\right\}\,n\mu_{1/n}(dx)\right]^n \\ &\approx \exp\left(\int \left\{e^{i\omega x} - 1\right\}\,n\mu_{1/n}(dx)\right) \text{ and also} \\ \mathsf{E}\left[e^{i\omega X}\right] &= \exp\left(\int \left\{e^{i\omega u} - 1\right\}\,\nu(du).\right)\end{aligned}$$

It follows (under suitable regularity conditions) that

$$n\mu_{1/n} \Rightarrow \nu \tag{9}$$

and often the pdfs converge. For example, if $\mu = \mathsf{NB}(\alpha, p)$ then $\mu_{1/n} = \mathsf{NB}(\alpha/n, p)$ and

$$\begin{aligned} n\ \mu_{1/n}(\{k\}) &= n\ \binom{\alpha/n + k - 1}{k} p^{\alpha/n} q^k \\ &= n\ \frac{\Gamma(k + \alpha/n)}{k!\ \Gamma(\alpha/n)} p^{\alpha/n} q^k \to (\alpha/k)\, q^k = \nu(\{k\}) \end{aligned}$$

as $n \to \infty$ or, for the Gamma $\mathsf{Ga}(\alpha, \beta)$ distribution,

$$\begin{aligned} n\ \mu_{1/n}(dx) &= n\ \frac{\beta^{\alpha/n}\ x^{(\alpha/n)-1}}{\Gamma(\alpha/n)} e^{-\beta x} \mathbf{1}_{\{x>0\}}\, dx \\ &\to \alpha\, x^{-1} e^{-\beta x} \mathbf{1}_{\{x>0\}}\, dx = \nu(dx) \end{aligned}$$

and for the Cauchy $\mathsf{Ca}(m, \gamma)$,

$$n\ \mu_{1/n}(dx) = n\ \frac{\gamma/n\pi}{(\gamma/n)^2 + (x - m/n)^2}\ dx \to \frac{\gamma}{\pi\ x^2}\ dx = \nu(dx).$$

While this argument is informal, and a little hard to tighten up, it's often a good way to work out what the Lévy measure is for some distribution you know (or just suspect) is ID. You can then confirm relation if necessary.

2 Constructions

2.1 ID Random Variables

Let ν be a positive measure on $\mathbb{R}^d$ that satisfies Eqn (3b) and let $H(du) \sim \mathsf{Po}\big(\nu(du)\big)$ be a Poisson random Borel measure on $(\mathbb{R}^d, \mathcal{B}^d)$ with mean ν. Then the function $f(u) = u$ lies in $\Psi_{0\wedge 1}$ so

$$H[f] = \int_{\mathbb{R}^d} u\, H(du) = \sum u_j \tag{10}$$

is well-defined, equal to the sum of the countably-many elements of the (random) support $\mathrm{spt}(H) = \{u_j\} \subset \mathbb{R}^d$ of $H(du)$. Its ch.f. is

$$\mathsf{E}\left[e^{i\omega H[f]}\right] = \exp\left\{\int_{\mathbb{R}^d} \left(e^{i\omega u} - 1\right) \nu(du)\right\},$$

so $H[f] \sim \mathsf{ID}(\nu)$. Thus we can construct random variables with any ID distribution satisfying Eqn (3b) as Poisson random integrals.

2.2 SII Stochastic Processes

But more is true. We can construct a **stochastic process** with *stationary independent increments* (SII) just as easily: let H be a Poisson random measure on the space $\mathcal{X} = \mathbb{R}^d \times \mathbb{R}_+$ with intensity measure $\nu(du)\,ds$, the product measure of $\nu(du)$ on $\mathbb{R}^d$ and and Lebesgue measure ds on $\mathbb{R}_+$, and set $f_t(u,s) := u\mathbf{1}_{\{0<s\le t\}}$; then for $0<r<t$

$$X_t \equiv H[f_t] = \sum \{u_j :\ 0 < s_j \le t\}$$

(where $\{(u_j,s_j)\} = \text{spt}(H)$) satisfies

$$\begin{aligned}\log \mathsf{E}\big[e^{i\omega(X_t-X_r)}\big] &= \int_{\mathbb{R}^d\times\mathbb{R}_+} \big(e^{i\omega[f_t-f_r]}-1\big)\,\nu(du)\,ds \\ &= \int_{\mathbb{R}^d\times\mathbb{R}_+} \big(e^{i\omega u}-1\big)\mathbf{1}_{\{r<s\le t\}}\,\nu(du)\,ds \\ &= (t-r)\int_{\mathbb{R}^d} \big(e^{i\omega u}-1\big)\,\nu(du).\end{aligned}$$

Thus X_t is an SII process with $X_0 = 0$ and $(X_t - X_r) \sim \mathsf{ID}\big((t-r)\nu\big)$.

2.3 SII Random Fields & Stochastic Integrals

Even more generally— if $m(d\sigma)$ is a σ-finite measure on any measurable space $(\Sigma,\mathcal{F})$, we can construct a d-dimensional ID-valued random measure $\Gamma(d\sigma)$ on Σ by letting $H \sim \mathsf{Po}\big(\nu(du)m(d\sigma)\big)$ be a Poisson random measure on $\mathbb{R}^d \times \Sigma$ with product intensity measure $\nu \otimes m$ and setting

$$\Gamma(A) = H\big[u\mathbf{1}_A(\sigma)\big] = \sum \{u_j :\ \sigma_j \in A\}$$

for each $A \in \mathcal{F}$ with $m(A) < \infty$. By linearity and continuity, this determines stochastic integrals of suitable functions $\phi(\sigma)$ on Σ, *i.e.*, those $\phi : \Sigma \to \mathbb{R}$ for which the function $f(u,\sigma) := u\phi(\sigma)$ is in $\Psi_{0\wedge 1}$:

$$\Gamma[\phi] = \int_{\mathbb{R}^d\times\Sigma} u\,\phi(\sigma)H(du\,d\sigma) = \sum \{u_j\phi(\sigma_j)\}$$

where $\{(u_j,\sigma_j)\} = \text{spt}(H)$. In this way we can construct stochastic integrals of random measures that assign independent random variables $\Gamma(A_i) \sim \mathsf{ID}\big(m(A_i)\nu(du)\big)$ to disjoint sets $\{A_i\}$ with ID distributions from any ID family (Gamma, Negative Binomial, SαS with $\alpha < 1$, *etc.*) that satisfies Eqn (3b). In Section (3) we consider going beyond that condition.

2.4 Truncation Algorithm

In Section (2.2) the SII process $X_t \sim \mathsf{ID}(t\nu)$ with Lévy measure ν on $\mathbb{R}^d$ satisfying Eqn (3b) was expressed in the form

$$X_t \equiv H[f_t] = \sum \{u_j : \ 0 < s_j \le t\} \tag{11}$$

for $f_t(u,s) := u\mathbf{1}_{\{0<s\le t\}}$ and $H \sim \mathsf{Po}\big(\nu(du)\,ds\big)$. Since $\mathrm{spt}(H) = \{(u_j, s_j)\}$ is infinite whenever $\nu(\mathbb{R}^d) = \infty$ (the usual case), some sort of approximation is necessary for any practical implementation and some sort of convergence must be established. In this section we present one such method and study its convergence.

For each $\epsilon > 0$ let B_ϵ denote the ball of radius ϵ in $\mathbb{R}^d$, and B_ϵ^c its complement. Since

$$\epsilon\mathbf{1}_{\{|u|>\epsilon\}} \le \big(1 \wedge |u|\big)$$

for all $u \in \mathbb{R}^d$, Eqn (3b) implies that $\nu(B_\epsilon^c) < \infty$ and so that the measure

$$\nu_\epsilon(du) \equiv \mathbf{1}_{\{|u|>\epsilon\}}\nu(du)$$

on $\mathbb{R}^d$ is finite. If $\nu_\epsilon^+ \equiv \nu(B_\epsilon^c) > 0$ then $\nu_\epsilon(du)/\nu_\epsilon^+$ is a probability measure on $\mathbb{R}^d$. One approach to constructing X_t explicitly on an interval $0 < t \le T$ is to replace f_t with $f_t^\epsilon := u\mathbf{1}_{\{|u|>\epsilon\}}\mathbf{1}_{\{0<s\le t\}}$ in Eqn (11):

Truncation Algorithm:

1. Fix $\epsilon > 0$;
2. Draw $J_\epsilon \sim \mathsf{Po}\big(T\nu_\epsilon^+\big)$;
3. Draw J_ϵ iid variates $\{s_j\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,T)$ and $\{u_j\} \overset{\text{iid}}{\sim} \nu_\epsilon(du)/\nu_\epsilon^+$;
4. Set
$$X_t^\epsilon := \sum \{u_j : \ 0 < s_j \le t\} \tag{12}$$
for $0 \le t \le T$. The sum has at most $J_\epsilon < \infty$ terms.

Truncation Error Estimates

The truncation error satisfies

$$(X_t - X_t^\epsilon) = H[f_t] - H[f_t^\epsilon] \ = H[f_t - f_t^\epsilon]$$

with L_1 norm bounded for $0 \le t \le T$ by

$$\mathsf{E}|X_t - X_t^\epsilon| = t\Big|\int_{B_\epsilon} u\nu(du)\Big| \ \le T\int_{\mathbb{R}^d} |u|\mathbf{1}_{\{|u|\le\epsilon\}}\nu(du)$$

which tends to zero as $\epsilon \to 0$ uniformly in $t \le T$ by Lebesgue's DCT, since $|u\mathbf{1}_{\{|u|\le\epsilon\}}| \le (1 \vee \frac{1}{\epsilon})(1 \wedge |u|) \in L_1(\mathbb{R}^d, \mathcal{B}^d, d\nu)$ and $|u\mathbf{1}_{|u|\le\epsilon}| \le \epsilon \to 0$.

If we set $\mu_\epsilon \equiv \int_{B_\epsilon} u\nu(du)$, then $M_t^\epsilon \equiv [X_t - X_t^\epsilon - t\mu_\epsilon]$ is a square-integrable $\mathbb{R}^d$-valued martingale with quadratic variation (see Appendix A)

$$[M]_t = \int_{B_\epsilon \times (0,t]} uu' \, H(du\, ds)$$

and predictable quadratic variation $\langle M \rangle_t = t\Sigma_\epsilon$ where

$$\Sigma_\epsilon \equiv \int_{B_\epsilon} uu' \, \nu(du).$$

The covariance is also $\mathsf{Cov}(X_t - X_t^\epsilon) = \mathsf{E} M_t M_t' = t\Sigma_\epsilon$. For any $\omega \in \mathbb{R}^d$, $\omega' M_t$ is a one-dimensional L_2 martingale and, by the martingale maximal inequality (34c),

$$\begin{aligned} \mathsf{E}\left[\sup_{0<s\le t} |\omega' M_s|^2\right] &= \mathsf{E}\left[\sup_{0<s\le t} |\omega'(X_s - X_s^\epsilon - s\mu_\epsilon)|^2\right] \\ &\le 4\mathsf{E}|\omega' M_t|^2 = 4t\ \omega'\Sigma_\epsilon\omega, \text{ and so} \\ \mathsf{E}\left[\sup_{0<s\le t} |\omega'(X_s - X_s^\epsilon)|^2\right] &\le t^2|\omega'\mu_\epsilon|^2 + 4t\omega'\Sigma_\epsilon\omega \end{aligned}$$

or, summing over unit vectors ω in the coordinate directions in $\mathbb{R}^d$,

$$\mathsf{E}\left[\sup_{0<s\le t} |X_s - X_s^\epsilon|^2\right] \le t^2|\mu_\epsilon|^2 + 4t\,\mathrm{tr}\big(\Sigma_\epsilon\big).$$

Pointwise bounds are also available, such as

$$\mathsf{E}|X_t - X_t^\epsilon|^2 = t^2|\mu_\epsilon|^2 + t\,\mathrm{tr}(\Sigma_\epsilon).$$

Example: α-Stable

For example, in $d = 1$ dimension, let $X_t \sim \mathsf{St_A}(\alpha, \beta, \gamma t, \delta t)$ be the α-stable SII process for some $0 < \alpha < 1$. Then

$$\begin{aligned} \mu_\epsilon = \int_{-\epsilon}^{\epsilon} u\,\nu(du) &= \int_{-\epsilon}^{\epsilon} u\,\gamma c_\alpha (1 + \beta\,\mathrm{sgn}\,u)\alpha|u|^{-\alpha-1}\,du \\ &= 2\alpha\beta\gamma c_\alpha \int_0^\epsilon u^{-\alpha}\,du = \frac{2\alpha\beta\gamma c_\alpha}{1-\alpha}\ \epsilon^{1-\alpha} \end{aligned}$$

(note this diverges as $\alpha \nearrow 1$ and would be $\pm\infty$ for $\alpha \geq 1$), and

$$\begin{aligned}\Sigma_\epsilon = \int_{-\epsilon}^{\epsilon} u^2\,\nu(du) &= \int_{-\epsilon}^{\epsilon} u^2\,\gamma c_\alpha(1+\beta\,\mathrm{sgn}\,u)\alpha|u|^{-\alpha-1}\,du \\ &= 2\alpha\gamma c_\alpha \int_0^\epsilon u^{1-\alpha}\,du = \frac{2\alpha\gamma c_\alpha}{2-\alpha}\;\epsilon^{2-\alpha}.\end{aligned}$$

For $\alpha = \frac{1}{2}$, $\beta = 1$, and $\gamma = 1$ (the Inverse Gaussian process), $\mu_\epsilon = \sqrt{2\epsilon/\pi}$ and $\Sigma_\epsilon = \sqrt{2\epsilon^3/9\pi}$ so the pointwise bounds are

$$\mathsf{E}|X_t - X_t^\epsilon|^2 \leq \frac{2\epsilon t^2}{\pi} + t\;\sqrt{2\epsilon^3/9\pi}$$

and the L_2 martingale bounds

$$\begin{aligned}\mathsf{E}\left[\sup_{0<s\leq t}\left|X_s - X_s^\epsilon - s\sqrt{2\epsilon/\pi}\right|^2\right] &\leq 4t\;\sqrt{2\epsilon^3/9\pi} \\ \mathsf{E}\left[\sup_{0<s\leq t}\left|X_s - X_s^\epsilon\right|^2\right] &\leq \frac{2\epsilon t^2}{\pi} + 4t\;\sqrt{2\epsilon^3/9\pi}.\end{aligned}$$

Exceedance probability bounds (34a) are available as well, such as:

$$\mathsf{P}\left[\sup_{0<s\leq t}\left\{X_s - X_s^\epsilon - s\sqrt{2\epsilon/\pi}\right\} \geq \lambda\right] \leq \lambda^{-1}\mathsf{E}M_t^+ \leq t\lambda^{-1}\sqrt{8\epsilon/\pi}.$$

In applications, these may be used to help inform the choice of $\epsilon > 0$ in a trade-off of accuracy *vs.* computational complexity (see Section (3.5.1)).

2.5 The ILM Algorithm

One shortcoming of the Truncation Algorithm is that ϵ is specified at the outset. In some problems it may be preferable to balance the costs of overly coarse approximations (ϵ too large) *vs.* computational complexity (ϵ too small) dynamically. The "Inverse Lévy Measure" (or ILM) algorithm introduced in (Wolpert and Ickstadt 1998a,b) generates the mass points $\{(u_n, s_n)\}$ of H successively in *decreasing order* of $|u_n|$, supporting dynamic stopping rules.

The starting point is the recognition that, as a function of $r > 0$ *decreasing* from $r = \infty$ down to $r = 0$, the increments

$$H\big(B_u^c \times (0,T]\big) - H\big(B_r^c \times (0,T]\big) = H\big((B_r \backslash B_s) \times (0,T]\big)$$

for $\infty > r > u \geq 0$ are the independent Poisson random variables that H assigns to disjoint annular cylinders— so $H(B_r^c \times (0,T])$ is an independent-increment Poisson process in the (*de*creasing) r variable and may be written in the form $H\big(B_r^c \times (0,T]\big) = P\big(T\nu_r^+\big)$ for some non-increasing function $r \mapsto \nu_r^+ : \mathbb{R}_+ \to \mathbb{R}_+$ and a standard unit-rate Poisson process $P(\cdot)$. By matching means,

$$T\nu_r^+ = \mathsf{E}\big[H(B_r^c \times (0,T])\big] = T\nu\big(B_r^c\big)$$

so $\nu_r^+ = \nu\big(B_r^c\big)$ and the successively smaller moduli $r_n \equiv |u_n|$ of the mass points $\{(u_n, s_n)\}$ may be generated by taking the event times $\{\tau_n\}$ of the unit-rate Poisson process and setting

$$r_n := \sup\big\{r \geq 0: \; T\nu_r^+ \geq \tau_n\big\}.$$

Now we turn to the angular part $\sigma_n = u_n/|u_n|$. Write the measure $\nu(dx)$ in polar form on $(0,\infty) \times S^{d-1}$ as the semidirect product

$$\nu(du) = \nu_r(dr)\; \nu_\sigma(d\sigma \mid r)$$

of a σ-finite positive measure ν_r on $\mathbb{R}_+$ and a "regular conditional probability distribution" (rcpd) $\nu_\sigma(d\sigma \mid r)$ on $S^{d-1} \times (0,\infty)$ (an rcpd is a kernel such that for each fixed Borel $A \subset S^{d-1}$, the function $r \rightsquigarrow \nu_\sigma(A \mid r)$ is Borel measurable and, for each fixed $r > 0$, $\nu_\sigma(\cdot \mid r)$ is a Borel probability measure on S^{d-1}). The radial measure $\nu_r(dr)$ is determined uniquely by $\nu_r\big((r,\infty)\big) = \nu_r^+ \equiv \nu(B_r^c)$, while the existence and uniqueness of ν_r are ensured by Rokhlin's (1949) "disintegration theorem" (Hoffmann-Jørgensen 1971) since $\mathbb{R}_+$ and S^{d-1} are Polish and hence Radon spaces.

If ν is absolutely-continuous wrt Lebesgue measure in $\mathbb{R}^d$ with density function $\nu(u)$, then ν_r and ν_σ will also be absolutely continuous, with

$$\nu_r(dr) = \left\{\int_{S^{d-1}} \nu(r\tilde{\sigma})\, d\tilde{\sigma}\right\} \frac{2\pi^{d/2}}{\Gamma(d/2)}\, r^{d-1}\, dr \quad \nu_\sigma(d\sigma \mid r) = \frac{\nu(r\sigma)}{\int_{S^{d-1}} \nu(r\tilde{\sigma})\, d\tilde{\sigma}}\, d\sigma$$

where dr denotes Lebesgue measure on $\mathbb{R}_+$ and $d\sigma$ is uniform measure on S^{d-1}, respectively, but (since $\mathbb{R}^+$ and S^{d-1} are Radon spaces) such a decomposition is available for *any* σ-finite ν, even if not absolutely continuous. In most of our applications $\nu_\sigma(d\sigma \mid r)$ will not depend on r.

The SII process $X_t \sim \mathsf{ID}(t\nu)$ can now be constructed using the ILM algorithm from successive Poisson event times $\{\tau_n\}$ as

$$X_t = \sum_{n<\infty} u_n \mathbf{1}_{\{0<s_n\leq t\}} \tag{13}$$

where $u_n = r_n \sigma_n$ for $n \in \mathbb{N}$ with

$$r_n = \sup\left\{r \ge 0 :\ T\nu_r^+ \ge \tau_n\right\} \quad \sigma_n \overset{\text{ind}}{\sim} \nu_\sigma(d\sigma \mid r_n) \quad s_n \overset{\text{iid}}{\sim} \mathsf{Un}(0,T).$$

For finite Lévy measures with $\nu_0^+ < \infty$, the number of terms $N \sim \mathsf{Po}(T\nu_0^+)$ in Eqn (13) is finite, and $r_n = 0$ for all $n > N$ since $T\nu_r^+ < \tau_n$ for all $r \ge 0$.

2.5.1 Example: Gamma

The polar representation of the $\mathsf{Ga}(\alpha,\beta)$ Lévy measure $\nu(du) = \alpha u^{-1} e^{-\beta} \mathbf{1}_{\{u>0\}}$ is simply $\nu_r(dr) = \alpha r^{-1} e^{-\beta r} dr$ on $(0,\infty)$ and $\nu_\sigma(d\sigma) = \delta_1(d\sigma)$, a point mass at $+1 \in S^0 = \{-1,+1\}$. The $\{r_n\}$ are determined by the relation

$$\tau_n = T\nu_{r_n}^+ = T\int_{r_n}^\infty \alpha u^{-1} e^{-\beta u}\, du = \alpha T \mathrm{E}_1(\beta r_n)$$

for the exponential integral function $\mathrm{E}_1(z) \equiv \int_z^\infty e^{-t} t^{-1} dt$ (Abramowitz and Stegun 1964, *p.* 218), so $X_t = \sum u_n \mathbf{1}_{\{s_n \le t\}}$ with $u_n = r_n$ given by

$$u_n = \mathrm{E}_1{}^{-1}(\tau_n/\alpha T)/\beta.$$

The exponential integral function E_1 is included as `gsl_sf_expint_E1()` in the GSL scientific library and its inverse can be computed by Newton's method; alternately, both E_1 and $\mathrm{E}_1{}^{-1}$ may be approximated very well within `R` as limits of the complimentary CDF and quantile functions of the gamma distribution as the shape parameter $\alpha \to 0$:

```
E1    <- function(x, alp=1e-9) {pgamma(x, alp, lower=F)/alp};
E1inv <- function(y, alp=1e-9) {qgamma(alp*y, alp, lower=F)};
```

2.5.2 Example: Poisson

The ILM algorithm works even for discrete Lévy measures. For the Poisson with rate $\lambda > 0$, for example, $\nu(du) = \lambda\delta_1(du)$ and so $\nu_r(dr) = \lambda\delta_1(dr)$ and $\nu_\sigma(d\sigma) = \delta_1(d\sigma)$, so $\nu_r^+ = \lambda$ for $r < 1$ and zero for $r \ge 1$. Thus $\sigma_n \equiv 1$ and

$$u_n = r_n = \sup\left\{r > 0 : T\lambda \mathbf{1}_{\{r<1\}} \ge \tau_n\right\}$$

is one if $\tau_n \le \lambda T$ and otherwise zero, *i.e.*, is one for $n \le N$ where

$$N \equiv \#\{n :\ \tau_n \le \lambda T\} \sim \mathsf{Po}(\lambda T).$$

The number of those with $s_n \le t$ (where $\{s_n\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,T)$),

$$X_t = \sum u_n \mathbf{1}_{\{s_n \le t\}} = \#\{n \le N :\ s_n \le t\},$$

is the sum of N iid Bernoulli random variables with means t/T, so it has a $\mathsf{Po}(\lambda t)$ distribution as required.

2.5.3 Example: α-Stable

The Lévy measure density $\nu(u) = \nu(-u) = \gamma c_\alpha \alpha |u|^{-\alpha-1}$ for the *symmetric* α-stable distribution $\mathsf{St_A}(\alpha, 0, \gamma, 0) = \mathsf{ID}(\nu)$ was given in Eqn (8). More generally, for $-1 \le \beta \le 1$ the asymmetric or *skewed* (if $\beta \ne 0$) α-stable distribution $\mathsf{St_A}(\alpha, \beta, \gamma, 0)$ is $\mathsf{ID}(\nu)$ for Lévy measure $\nu(du) = \nu(u)\,du$ with density function

$$\nu(u) = \gamma c_\alpha \alpha \big(1 + \beta \operatorname{sgn} u\big)\, |u|^{-\alpha-1} \tag{14}$$

proportional to $|u|^{-\alpha-1}$ for all u, but with different coefficients for $u > 0$ and $u < 0$. The polar representation of $\nu(du)$ is a product measure $\nu(du) = \nu_r(dr)\nu_\sigma(d\sigma)$ with $\nu_r^+ = 2\gamma c_\alpha r^{-\alpha}$ and hence

$$\begin{aligned} \nu_r(dr) &= 2\gamma c_\alpha \alpha r^{-\alpha-1}\, dr, \qquad r > 0 \\ \nu_\sigma(\{\sigma\}) &= (1 + \beta\sigma)/2, \qquad \sigma \in S^0 = \{\pm 1\} \end{aligned}$$

and so the ILM construction sets

$$X_t = \sum u_n \, \mathbf{1}_{\{s_n \le t\}} \tag{15}$$

with $u_n = r_n \sigma_n$ and

$$r_n = (2\gamma T c_\alpha / \tau_n)^{1/\alpha} \qquad \sigma_n = (2Z_n - 1) \qquad s_n \stackrel{\text{iid}}{\sim} \mathsf{Un}(0, T)$$
$$Z_n \stackrel{\text{iid}}{\sim} \mathsf{Bi}\big(1, \tfrac{1+\beta}{2}\big)$$

with random signs $\sigma_n = \pm 1$ (here expressed in terms of Bernoullis Z_n). The ch.f. (21) is found for this distribution in Section (3.3). Typically $\tau_n \approx n \pm \sqrt{n}$ and hence $r_n \asymp n^{-1/\alpha} \pm n^{-2/\alpha}$, so we cannot expect the sum in Eqn (15) to converge absolutely for $\alpha \ge 1$. For the **Cauchy**, for example, with $\alpha = 1$ and $c_\alpha = 1/\pi$,

$$X_t = \frac{2\gamma T}{\pi} \sum_{n=1}^{\infty} \frac{\sigma_n}{\tau_n}$$

(with $\sigma_n = \pm 1$ with probabilities $\frac{1}{2}$ each) converges in L_2, but not absolutely.

3 Compensation

Again let $\big(\mathcal{X}, \mathcal{F}, \nu(dx)\big)$ be a σ-finite measure space, and $H(dx) \sim \mathsf{Po}(\nu)$ a random measure assigning independent Poisson-distributed random variables $H(A_n) \sim \mathsf{Po}\big(\nu(A_n)\big)$ to disjoint sets $A_n \in \mathcal{F}$ of finite measure $\nu(A_n) <$

∞. Define the (fully) *Compensated* Poisson measure to be

$$\tilde{H}(A) := H(A) - \nu(A)$$

for any set $A \in \mathcal{F}$ with finite measure $\nu(A) < \infty$, simply $H(A)$ minus its mean. Again the stochastic integral of simple functions

$$f = \sum a_n \mathbf{1}_{\{A_n\}}$$

has to be

$$\begin{aligned}\tilde{H}[f] &:= \int_{\mathcal{X}} f(x)\, \tilde{H}(dx) = \sum a_n [H(A_n) - \nu(A_n)] \\ &= \sum f(x_n) - \int f(x) \nu(dx)\end{aligned}$$

for $\{x_n\} = \mathrm{spt}(H)$ and again the log ch.f. can be computed easily:

$$\begin{aligned}\log \mathsf{E}\big[e^{i\omega \tilde{H}[f]}\big] &= \sum_n \big[e^{i\omega a_n} - 1 - i\omega a_n\big]\, \nu(A_n) \\ &= \int_{\mathcal{X}} \Big[e^{i\omega f(x)} - 1 - i\omega f(x)\Big]\, \nu(dx). \end{aligned} \tag{16}$$

Again $f \mapsto H[f]$ can be extended by continuity to $L_1(\mathcal{X}, \mathcal{F}, \nu)$, and this time also to $L_2(\mathcal{X}, \mathcal{F}, \nu)$ (the mapping is an isometry from L_2 into the space of square-integrable random variables). Again we can do more.

3.1 Musielak-Orlicz spaces II

This time we can extend the definition of $\tilde{H}[f]$ to limits f of simple functions provided the integral in (16) is well-defined. The "compensation" (subtracting the mean of H) has changed the requirements on f, however, in two ways: it is *less* restrictive in that the term in braces is now $O(f^2)$ near $f \approx 0$, so $\nu(dx)$ can be more singular near the zeros of f, but it is *more* restrictive in that the term in braces is no longer bounded, so $\nu(dx)$ mustn't grow too fast in places where f is unbounded. The exact requirement is that f must lie in the M-O space

$$\Psi_{1\wedge 2} := \left\{ f : \int_{\mathcal{X}} \big(|f(x)| \wedge |f(x)|^2\big)\, \nu(dx) < \infty \right\}. \tag{17}$$

This contains $L_1(\mathcal{X}, \mathcal{F}, \nu)$ and also $L_2(\mathcal{X}, \mathcal{F}, \nu)$, but it is larger than their union.

We're still a step or two away from being able to construct random variables and stochastic processes with Cauchy (and other α-Stable distributions with $\alpha \geq 1$) by a scheme analogous to Eqn (10). An attempt to set $X \stackrel{?}{:=} \tilde{H}[f]$ for $\nu(du) = (\gamma/\pi)u^{-2}du$ and $f(u) = u$ still fails, just as setting $X \stackrel{?}{:=} H[f]$ did, but for a different reason— $f \notin \Psi_{0\wedge 1}$ because $\big(1 \wedge |f(u)|\big) \asymp |u|$ isn't ν-integrable at zero, while $f \notin \Psi_{1\wedge 2}$ because $\big(|f(u)| \wedge |f(u)|^2\big) f(u) \asymp |u|$ isn't ν-integrable at infinity. What is needed is a way to "compensate" (by subtracting the Poisson mean) only near zero, and not at infinity.

3.2 Partial Compensation

Let ν be a σ-finite measure on $\mathbb{R}^d$ satisfying

$$\int_{\mathbb{R}^d} \big(1 \wedge u^2\big)\, \nu(du) < \infty \tag{4b}$$

and let $h : \mathbb{R}^d \to \mathbb{R}^d$ be a Borel function that is

- uniformly *bounded*, and satisfies
- $h(u) = u + O(|u|^2)$ near $u \approx 0$.

One common choice is $h_1(u) = u\mathbf{1}_{\{|u|<1\}}$, but for studying and constructing α-stable random variables and processes a better choice is $h(u) = \frac{u}{|u|}\sin|u|$ or, for $d = 1$, simply $h(u) = \sin u$. Heuristically what we would like to do now to generate $X \sim \mathsf{ID}(\nu)$ is to set $f(u) = u$ and look at

$$\int_{\mathbb{R}^d} uH(du) - \int_{\mathbb{R}^d} h(u)\nu(du),$$

but neither of those two integrals converges for ν that satisfy the local L_2 condition (4b) but *not* the local L_1 condition (3b) (for example, the SαS measure $\nu(du) = c_\alpha \alpha |u|^{-\alpha-1}\, du$ for $1 \leq \alpha < 2$). What *does* work (Wolpert and Taqqu 2005), consistent with that intuition, is to set $f(u) := u$ and

$$X = H[f - h] + \tilde{H}[h]. \tag{18}$$

This is well-defined because $|f(u) - h(u)|$ is $O(u^2)$ near zero and $O(|u|)$ near infinity so $[f - h] \in \Psi_{0\wedge 1}$ (so $H[f - h]$ is well-defined), while $h(u)$ is $O(|u|)$ near zero and bounded near infinity, so $h \in L_2(\mathcal{X}, \mathcal{F}, \nu) \subset \Psi_{1\wedge 2}$ (so $\tilde{H}[h]$ is well-defined). For functions $f, h \in L_1(\mathbb{R}, \mathcal{B}, \nu)$, (18) would be

$$\begin{aligned} X &= \int_{\mathbb{R}^d} [f(u) - h(u)] H(du) + \int_{\mathbb{R}^d} h(u)[H(du) - \nu(du)] \\ &= \int_{\mathbb{R}^d} f(u) H(du) - \int_{\mathbb{R}^d} h(u)\nu(du), \end{aligned}$$

with log ch.f.

$$\log \mathsf{E} e^{i\omega X} = \int_{\mathbb{R}^d} \left[e^{i\omega f(u)} - 1 - i\omega h(u) \right] \nu(du). \tag{19}$$

Eqn (19) remains true for all f and h for which the integral is well-defined, and in particular for $f(u) = u$. For the choice $h(u) = \sin u$ (or any other odd function) in $\mathbb{R}^1$ and ν of (8), $X = H[f - h] + \tilde{H}[h]$ has log ch.f.

$$\log \mathsf{E} e^{i\omega X} = \gamma c_\alpha \int_{\mathbb{R}} \left[e^{i\omega u} - 1 - i\omega \sin u \right] \alpha |u|^{-\alpha-1}\, du = -\gamma |\omega|^\alpha$$

for any $0 < \alpha < 2$, so we have succeeded in constructing SαS random variables with arbitrary shape parameters, including the Cauchy for $\alpha = 1$.

Once again we can extend this immediately to SII *processes* $X_t = H[f_t - h_t] + \tilde{H}[h_t]$ with increments $[X_t - X_s] \sim \mathsf{St_A}(\alpha, 0, \gamma|t-s|, 0)$ by setting $f_t(u, s) = u \mathbf{1}_{\{0<s\le t\}}$ and $h_t(u, s) = (\sin u)\mathbf{1}_{\{0<s\le t\}}$ on $\mathcal{X} = \mathbb{R} \times \mathbb{R}_+$ with $H \sim \mathsf{Po}\big(\nu(du)\, ds\big)$, or to S$\alpha$S-valued *random fields* $\Gamma(d\sigma) \sim \mathsf{St_A}(\alpha, 0, \gamma(d\sigma), 0)$ on any σ-finite measure space $\big(\Sigma, \mathcal{F}, \gamma\big)$ by setting $\Gamma[A] = H[f_A - h_A] + \tilde{H}[h_A]$ for $f_A(u, \sigma) = u\mathbf{1}_A(\sigma)$ and $h_A(u, \sigma) = \sin u \mathbf{1}_A(\sigma)$ with Poisson random measure $H \sim \mathsf{Po}\big(c_\alpha \alpha |u|^{-\alpha-1} du\, \gamma(d\sigma)\big)$ on $\mathbb{R} \times \Sigma$.

3.3 Skewed α-Stable Processes

Fix $0 < \alpha < 2$, $-1 \le \beta \le 1$, and $\gamma > 0$; set $c_\alpha = \frac{1}{\pi}\Gamma(\alpha) \sin \frac{\pi\alpha}{2}$ and

$$\nu(du) = \gamma c_\alpha \big(1 + \beta \operatorname{sgn} u\big)\, \alpha |u|^{-\alpha-1}\, du. \tag{20}$$

For $0 < \alpha < 1$ this satisfies Eqn (3b) so we can construct (as we did in Section (2.5.3)) random variables and processes with log ch.f.

$$\begin{aligned} \log \chi(\omega) &= \int_{\mathbb{R}} \big(e^{i\omega u} - 1\big) \nu(du) \\ &= c_- \int_{-\infty}^0 \big(e^{i\omega u} - 1\big)(-u)^{-\alpha-1}\, du + c_+ \int_0^\infty \big(e^{i\omega u} - 1\big)(u)^{-\alpha-1}\, du \\ &= c_- \int_0^\infty \big(e^{-i\omega u} - 1\big)(u)^{-\alpha-1}\, du + c_+ \int_0^\infty \big(e^{i\omega u} - 1\big)(u)^{-\alpha-1}\, du \\ &= -\gamma |\omega|^\alpha \left\{ 1 - i\beta \tan \tfrac{\pi\alpha}{2} \operatorname{sgn} \omega \right\}, \end{aligned}$$

where $c_\pm = \gamma c_\alpha (1 \pm \beta)\alpha$.

This ch.f. χ is, in fact, valid for all $\alpha \neq 1$ in the interval $(0, 2)$. Adding a location $\delta \in \mathbb{R}$ to a random variable with this ch.f. leads to one with ch.f.

$$\chi(\omega) = \exp\left[i\delta\omega - \gamma|\omega|^\alpha \left\{1 - i\beta \tan\tfrac{\pi\alpha}{2}\ \mathrm{sgn}\,\omega\right\}\right]. \tag{21}$$

This is the so-called "Zolotarev (A)" parametrization of the α-stable family of distributions (Zolotarev 1986, *pp.* 9), which we'll denote $\mathsf{St}_\mathsf{A}(\alpha, \beta, \gamma, \delta)$, with four parameters: *shape* α, *asymmetry* β, *intensity* γ, and *location* δ. There is a (pretty funny) checkered history of parametrizations of the α-stable distributions (Hall 1981). While everyone agrees on α (both the letter and its role), authors have disagreed about both letters and roles of the other three. Early authors got the sign of β "wrong" (in the sense that their $\beta > 0$ was associated with a $\nu(du)$ heaviest on the *negative* half-line). Many authors prefer a *scale* parameter σ rather than intensity; they're related by $\gamma = \sigma^\alpha$. And Zolotarev in fact uses $\gamma\delta$ as his location, while most other authors use δ (including me, for reasons we'll see later). In most arguments $\delta = 0$, where we can all agree.

The ch.f. Eqn (21) is well-defined and valid for all $0 < \alpha \leq 2$, $-1 \leq \beta \leq 1$, $\gamma \geq 0$, and $\delta \in \mathbb{R}$ *except* for $\alpha = 1$ with $\beta \neq 0$— where the tangent function approaches $+\infty$ as $\alpha \nearrow 1$, or $-\infty$ as $\alpha \searrow 1$, so χ can't be extended by continuity to $\alpha = 1, \beta \neq 0$ (at least, not directly).

3.3.1 Compensation: The (M) Parametrization

By an entertaining use of analytic continuation and the Gamma reflection identity, we can evaluate the log ch.f. for the partially-compensated α-stable distribution with Lévy measure ν given in Eqn (20), with an offset $\delta \in \mathbb{R}$:

$$\begin{aligned}\log \chi_\alpha(\omega) &= i\delta\omega + \int_\mathbb{R} \left[e^{i\omega u} - 1 - i\omega \sin u\right]\ \nu(du) \\ &= i\delta\omega - \gamma|\omega|^\alpha - i\beta\gamma\ \tan\tfrac{\pi\alpha}{2}\ \omega\big(1 - |\omega|^{\alpha-1}\big) \qquad \text{(22a)}\end{aligned}$$

for $\alpha \neq 1$. This expression *is* continuous, so can be extended by continuity to $\alpha = 1$ (by L'Hôspital's rule, for example):

$$\log \chi_1(\omega) = i\delta\omega - \gamma|\omega| \quad - i\beta\gamma\,\frac{2}{\pi}\,\omega \log|\omega| \tag{22b}$$

For $\delta = 0$ and $\beta = 0$ these reduce to the SαS solution $-\gamma|\omega|^\alpha$, but for $\beta \neq 0$ they're new. This continuous parametrization is called the "Zolotarev (M)" parametrization $\mathsf{St}_\mathsf{M}(\alpha, \beta, \gamma, \delta)$ of the α-stable family.[1]

[1] except that, again, Zolotarev used $\delta\gamma$ as location instead of δ.

By consolidating factors of ω in (22a), we see that the ch.f. for a random variable $X \sim \mathsf{St}_\mathsf{M}(\alpha,\beta,\gamma,\delta)$ can be re-written

$$\begin{aligned}\log \chi_\alpha(\omega) &= i\big(\delta - \beta\gamma\tan\tfrac{\pi\alpha}{2}\big)\omega - \gamma|\omega|^\alpha + i\beta\gamma\,\tan\tfrac{\pi\alpha}{2}\,\omega|\omega|^{\alpha-1}\\ &= i\delta^*\omega - \gamma|\omega|^\alpha\left\{1 - i\beta\,\tan\tfrac{\pi\alpha}{2}\,\operatorname{sgn}\omega\right\},\end{aligned}$$

the $\mathsf{St}_\mathsf{A}(\alpha,\beta,\gamma,\delta^*)$ ch.f. of Eqn (21) with a different location parameter $\delta^* = [\delta - \beta\gamma\tan\frac{\pi\alpha}{2}]$— effectively, the (M) parametrization adds an α-dependent constant offset $(\delta^* - \delta) = -\beta\,\gamma\,\tan\frac{\pi\alpha}{2}$ to X in such a way that the distribution becomes a continuous function of $\alpha \in (0,2)$. For $1 < \alpha < 2$, $\mathsf{E}X = \delta^*$ and we have simply subtracted the mean, but for $0 < \alpha \le 1$ the mean is undefined. For $\beta \neq 0$ the constant becomes infinite as $\alpha \to 1$, of sign $\operatorname{sgn}(\beta(\alpha-1))$ that reverses at $\alpha = 1$. Both St_A and St_M parametrizations are convenient, at different times— for example, for $\alpha < 1$ the fully-skewed $\mathsf{St}_\mathsf{A}(\alpha,1,\gamma,0)$ is a well-defined distribution taking only positive values $0 < X < \infty$, making St_A more convenient, while St_M is better for skewed stables with $\alpha \ge 1$, or any time α is close to one.

3.4 Important Example: Inverse Gaussian Distribution

The skewed stable SII processes $X_t \sim \mathsf{St}_\mathsf{A}(\alpha,1,\gamma t,0)$ for $0 < \alpha < 1$ are called *subordinators.* They have non-decreasing paths with independent increments. They are commonly used as random time-changes for other SII processes (Brownian motion, for example), since the composition is once again SII. The most important of these is the case $\alpha = \frac{1}{2}$ (sometimes called the *Wald* process), because (a) its pdf is available in closed form, and (b) it has a beautiful connection with the Wiener process.

For $\alpha = \frac{1}{2}$, $c_\alpha = 1/\sqrt{2\pi}$ and the Lévy measure of Eqn (20) is

$$\nu(du) = \frac{\gamma}{\sqrt{2\pi}} u^{-3/2}\mathbf{1}_{\{u>0\}}\,du. \tag{23}$$

Compensation is unnecessary and an increasing process $X_t \sim \mathsf{St}_\mathsf{A}(\frac{1}{2},1,\gamma t,0)$ for $0 < s \le T$ may be constructed with Lévy measure $t\nu(du)$ by the ILM algorithm as the absolutely convergent sum

$$X_t = \gamma T\sqrt{2/\pi}\,\sum_{j=1}^\infty \frac{1}{\sqrt{\tau_j}}\mathbf{1}_{\{s_j \le t\}}$$

where $\{\tau_j\}$ are the event times of a standard Poisson process and $\{s_j\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,T)$.

3.4.1 Brownian Hitting Times

Let $W(t)$ be a standard Wiener process starting at zero, and for $\theta > 0$ set

$$\tau_\theta := \inf\{t \ge 0 : \; W(t) \ge \theta\}.$$

We'll see below that $\tau_\theta < \infty$ almost-surely. By the strong Markov property, the process $W(t+\tau_\theta) - \theta$ is also a standard Wiener process; it follows that the stochastic process $\theta \rightsquigarrow \tau_\theta$ is SII. By Brownian scaling, for any real $c \neq 0$ the process $B(t) = W(c^2 t)/c$ is also a standard Wiener process, so for $c > 0$,

$$\begin{aligned}
\tau_\theta &= \inf\{t \ge 0 : \; B(t) \ge \theta\} \\
&= \inf\big\{t \ge 0 : \; W(c^2 t) \ge c\theta\big\} \\
&= c^{-2} \inf\{s \ge 0 : \; W(s) \ge c\theta\} \\
&\overset{d}{\equiv} \tau_{c\theta}/c^2 \quad \text{or, for } c = 1/\theta, \\
\tau_\theta &\overset{d}{\equiv} \theta^2 \tau_1.
\end{aligned} \tag{24}$$

By the SII property the log characteristic function $\psi_\theta(\omega)$ of τ_θ must satisfy $\psi_\theta(\omega) \equiv \theta\psi_1(\omega)$, and by (24) also $\psi_\theta(\omega) = \psi_1(\omega\theta^2)$, so

$$\psi_\theta(\omega) = \theta|\omega|^{\frac{1}{2}} \psi_1(\operatorname{sgn}\omega)$$

and τ_θ has an α-stable distribution with $\alpha = \frac{1}{2}$. Since it's positive and starts at zero, also $\beta = 1$ and $\delta = 0$. For that reason, $\mathsf{St_A}(\frac{1}{2}, 1, \gamma, 0)$ is called the "inverse Gaussian distribution". The CDF and pdf are available from the reflection principle— conditional on $\tau_\theta < t$, $B(s) \equiv [W(s+\tau_\theta) - \theta]$ is a standard Wiener process with $\mathsf{P}[B(s) > 0] = \frac{1}{2}$ for every $s > 0$, so for $t > 0$,

$$\begin{aligned}
\mathsf{P}[W(t) \ge \theta] &= \mathsf{P}[W(t) \ge \theta, \; \tau_\theta < t] \\
&= \mathsf{P}[W(t) \ge \theta \mid \tau_\theta < t] \; \mathsf{P}[\tau_\theta < t] \\
&= \mathsf{P}[B(t - \tau_\theta) \ge 0 \mid \tau_\theta < t] \; \mathsf{P}[\tau_\theta < t] \\
&= \tfrac{1}{2}\mathsf{P}[\tau_\theta < t] \quad \text{and so} \\
\mathsf{P}[\tau_\theta \le t] &= 2\mathsf{P}[W(t) \ge \theta] \qquad = 2\Phi\big(-\theta t^{-\frac{1}{2}}\big).
\end{aligned}$$

Since this tends to one as $t \to \infty$, $\tau_\theta < \infty$ a.s. as claimed for any $\theta \ge 0$. Taking a derivative, the pdf at $t > 0$ for τ_θ is

$$f_\theta(t) = \theta t^{-3/2} \phi\Big(\frac{\theta}{\sqrt{t}}\Big) \qquad = \frac{\theta}{\sqrt{2\pi}} t^{-3/2} e^{-\theta^2/2t},$$

making conventional likelihood-based inference possible.

Surprisingly, by a "tilting" argument (Steele 2000, Ch. 13), the hitting time $\tau \equiv \inf\{s \ge 0 : W(t) \ge \theta - \lambda t\}$ of a *linear* boundary is not much harder to find. For $\lambda \ge 0$ it has pdf

$$f_\theta(t) = \theta t^{-3/2} \phi\Big(\frac{\theta - \lambda t}{\sqrt{t}}\Big) = \frac{\theta}{\sqrt{2\pi}} t^{-3/2} e^{\theta\lambda - (\lambda^2 t + \theta^2/t)/2}, \tag{25}$$

a special case ($p = -\frac{1}{2}$, $a = \lambda^2$, $b = \theta^2$) of the GiG distribution discussed in Section (3.4.2) below. The probability $\mathsf{P}[\tau < \infty]$ of ever hitting the boundary is just the integral over $\mathbb{R}_+$ of f_θ in Eqn (25). This is one for $\lambda \ge 0$, but $e^{2\lambda\theta} < 1$ for $\lambda < 0$, so it is then possible that $W(t)$ will *never* exceed $\theta - \lambda t$, making τ infinite. For negative λ the *conditional* distribution of τ, given that it is finite, is proportional to f_θ.

For $\lambda > 0$ the mean is $\mathsf{E}\tau = \theta/\lambda < \infty$ and the characteristic function is

$$\mathsf{E}[e^{i\omega\tau}] = \exp\Big\{\theta\lambda - \theta\sqrt{\lambda^2 - 2i\omega}\Big\},$$

while for $\lambda = 0$ the mean is infinite and the ch.f. reduces to

$$= \exp\Big\{-\theta|\omega|^{\frac{1}{2}}(1 - i\,\mathrm{sgn}\,\omega)\Big\}.$$

In both cases it is of course not real, because the distribution of τ is not symmetric about zero.

3.4.2 Generalized Inverse Gaussian Distribution

The Generalized Inverse Gaussian $\mathsf{GiG}(p; a, b)$ distribution for $a \ge 0$, $b \ge 0$, $p \in \mathbb{R}$ has pdf

$$f(x \mid p, a, b) = \frac{(a/b)^{p/2}}{2K_p\big(\sqrt{ab}\big)}\; x^{p-1}\; e^{-(ax+b/x)/2}\mathbf{1}_{\{x>0\}} \tag{26}$$

where $K_p(z)$ is a modified Bessel function of the third kind (Watson 1944, *p.* 185), available in `R` as `bessselK()`:

$$\begin{aligned} K_{\pm\nu}(z) &= \frac{\Gamma(\nu+\frac{1}{2})\;(2z)^\nu}{\sqrt{\pi}} \int_0^\infty \frac{\cos t}{(t^2+z^2)^{\nu+\frac{1}{2}}}\,dt \qquad \nu \ge 0 \\ &= \tfrac{1}{2}\int_0^\infty x^{\nu-1}\, e^{-z(x+x^{-1})/2}\,dx. \end{aligned}$$

For half-odd-integer ν the integral can be evaluated in closed form (Abramowitz and Stegun 1964, §10.2.17):

$$K_\nu(z) = \begin{cases} e^{-z}\sqrt{\pi/2z} & \nu = 1/2 \\ e^{-z}\sqrt{\pi/2z}\ (1+z^{-1}) & \nu = 3/2 \\ e^{-z}\sqrt{\pi/2z}\ (1+3z^{-1}+3z^{-2}) & \nu = 5/2 \\ e^{-z}\sqrt{\pi/2z}\ \sum_{0\le k<\nu}^{n}\binom{\nu}{k}(2z)^{-k} & \nu\in\mathbb{N}_0+\frac{1}{2} \end{cases} \tag{27}$$

For $p=-\frac{1}{2}$ the GiG (26) reduces to the Inverse Gaussian (and to the Wald distribution $\mathsf{St_A}(\frac{1}{2},1,\gamma,0)$ for $a=0$ and $b=\gamma^2$); for $b=0$ and $a>0$ it is just the Gamma $\mathsf{Ga}(p,a/2)$ distribution (note $z^\nu K_\nu(z)\to 2^{\nu-1}\Gamma(\nu)$ as $z\to 0$). For all parameter values it is ID, with ch.f. (see Jørgensen (1982) or Seshadri (1999))

$$\chi(\omega) = a^{p/2}(a-2i\omega)^{-p/2}K_p\big(\sqrt{b(a-2i\omega)}\big)\Big/K_p\big(\sqrt{ab}\big)$$

or, for $p=-\frac{1}{2}$,

$$\begin{aligned} f(x\mid a,b) &= \sqrt{b/2\pi}\ x^{-3/2}\ \exp\Big\{\sqrt{ab}-(ax+b/x)/2\Big\}, \quad x>0 \\ \chi(\omega) &= \exp\Big\{\sqrt{b}\big(\sqrt{a}-\sqrt{a-2i\omega}\big)\Big\}. \end{aligned} \tag{28}$$

For $a=0$ this is the α-stable ch.f.

$$\chi(\omega) = \exp\Big\{-\sqrt{2b}(\omega e^{-i\pi/2})^{\frac{1}{2}}\Big\} = \exp\Big\{-\sqrt{b}|\omega|^{\frac{1}{2}}\big(1-i\,\mathrm{sgn}\,\omega\big)\Big\},$$

ch.f. of the fully-skewed $\mathsf{St_A}(\frac{1}{2},1,\sqrt{b},0)$ distribution with $\gamma=\sqrt{b}$. After noting this is a convolution semigroup in the parameter $\gamma=\sqrt{b}$, the method of Section (1.3) can be applied to find the Lévy measure density for $p=-\frac{1}{2}$:

$$\begin{aligned} \nu(u) &= \lim_{n\to 0} n\,\frac{(a/b)^{-1/4}\ n^{-\frac{1}{2}}}{2K_{\frac{1}{2}}\big(\sqrt{ab}/n\big)}u^{-3/2}e^{-(au+b/n^2u)/2}\mathbf{1}_{\{u>0\}} \\ &= \sqrt{b/2\pi}\ u^{-3/2}e^{-au/2}\mathbf{1}_{\{u>0\}}, \end{aligned}$$

reducing for $a=0$ and $b=\gamma^2$ to $\gamma c_\alpha \alpha|u|^{-\alpha-1}\mathbf{1}_{\{u>0\}}$ for $\alpha=\frac{1}{2}$, as in Eqn (23) for the $\mathsf{St_A}(\frac{1}{2},1,\gamma,0)$ distribution.

If $X\sim\mathsf{GiG}(p;a,b)$ then $X^{-1}\sim\mathsf{GiG}(-p;b,a)$, so the family is closed under the operation of multiplicative inverse. It is also a three-parameter exponential family, with canonical parameter $(p,-a/2,-b/2)$ and sufficient statistic

$$T(\vec{t}) = \Big(\sum\log t_j,\ \sum t_j,\ \sum t_j^{-1}\Big).$$

3.5 Truncation Algorithm with Compensation

Compensation will enable us to construct an SII process whose Lévy measure satisfies the local-L_2 condition (4b), but perhaps not the local-L_1 condition (3b). Fix a bounded compensator function $h(u) = u + O(|u|^2)$ and consider the problem of constructing an approximation to a process X_t for which X_1 has the ch.f. of Eqn (4a) (with $m = 0$ and $\Sigma = 0$).

The truncation algorithm of Section (2.4) gets just one new step: we must subtract an ϵ-dependent offset. The most obvious choice for the offset would be $t\int_{B_\epsilon^c} h(u)\,\nu(du)$, but this is difficult to compute exactly in examples so instead we do a variation on this.

For $\epsilon > 0$ the function $h_\epsilon : \mathbb{R}^d \to \mathbb{R}^d$ given by $h_\epsilon(u) := u\mathbf{1}_{\{|u|\leq\epsilon\}}$ is a valid compensator. Thus the difference $h - h_\epsilon$ is bounded and $O(|u|^2)$ near zero, and so is ν-integrable by (4b). Set

$$\mu_\epsilon := \int_{\mathbb{R}^d} \big[h(u) - h_\epsilon(u)\big]\,\nu(du) \tag{29a}$$

This differs from $\int_{B_\epsilon^c} h(u)\,\nu(du)$ only by $\int_{B_\epsilon}[h(u) - u]\,\nu(du) = o(\epsilon)$. Now construct an approximation X_t^ϵ to X^t by:

Truncation Algorithm with Compensation:

1. Fix $\epsilon > 0$, and set $\nu_\epsilon^+ := \nu\big(B_\epsilon^c\big)$;
2. Fix $T > 0$, and draw $J_\epsilon \sim \mathsf{Po}\big(T\nu_\epsilon^+\big)$;
3. Draw J_ϵ iid variates $\{s_j\} \overset{\text{iid}}{\sim} \mathsf{Un}(0, T)$ and $\{u_j\} \overset{\text{iid}}{\sim} \nu_\epsilon(du)/\nu_\epsilon^+$;
4. For $0 \leq t \leq T$, set

$$X_t^\epsilon := \sum_{j\leq J_\epsilon} \{u_j :\; 0 < s_j \leq t\} - t\,\mu_\epsilon \tag{29b}$$

where μ_ϵ was given in (29a).

3.5.1 Truncation Error Estimates

The truncation error can be evaluated from

$$\begin{aligned} X_t &= H[(u-h)\mathbf{1}_{\{s\leq t\}}] \quad + \quad \tilde{H}[h\mathbf{1}_{\{s\leq t\}}] \\ &= H[(u-h_\epsilon)\mathbf{1}_{\{s\leq t\}}] \quad + \quad \tilde{H}[h_\epsilon\mathbf{1}_{\{s\leq t\}}] \quad + \quad t\int [h_\epsilon - h]\nu(du) \end{aligned}$$

while

$$X_t^\epsilon = H[(u-h_\epsilon)\mathbf{1}_{\{s\leq t\}}] \qquad\qquad + \quad t\int [h_\epsilon - h]\nu(du)$$

so the truncation error

$$M_t = \big[X_t - X_t^\epsilon\big] = \tilde{H}[h_\epsilon \mathbf{1}_{\{s\le t\}}]$$

is the L_2 martingale $M_t = \big[X_t - X_t^\epsilon\big]$ with quadratic variation

$$\begin{aligned}
[M]_t &= \int_{B_\epsilon\times(0,t]} uu'\,H(du\,ds)\\
\langle M\rangle_t &= t\Sigma_\epsilon \quad \text{for}\\
\Sigma_\epsilon &= \int_{B_\epsilon} uu'\,\nu(du).
\end{aligned}$$

As before, we have bounds such as

$$\mathsf{E}\big[\sup_{0<s\le t} |X_s - X_s^\epsilon|^2\big] \le 4\,t\,\mathrm{tr}\big(\Sigma_\epsilon\big).$$

3.5.2 Specific Examples

1. Skewed α-Stable

Consider $X_t \sim \mathsf{St}_\mathsf{M}(\alpha,\beta,\gamma t,\delta t)$ with Lévy measure $\nu(du) = \gamma c_\alpha \alpha\big(1 + \beta\,\mathrm{sgn}\,u\big)\,|u|^{-\alpha-1}\,du$ of Eqn (14). With $h(u)=\sin u$ and $c_\alpha = \frac{1}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}$, we have

$$\begin{aligned}
\mu_\epsilon &= \int \big[\sin u - u\mathbf{1}_{B_\epsilon}\big]\;\nu(du)\\
&= 2\alpha\beta\gamma c_\alpha\left\{\int_0^\epsilon [\sin(u)-u]\,u^{-\alpha-1}du + \int_\epsilon^\infty \sin(u)\,u^{-\alpha-1}du\right\}\\
&= \begin{cases} \frac{2\beta\gamma\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi}\left[\frac{\alpha\epsilon^{1-\alpha}}{\alpha-1} + \Gamma(1-\alpha)\sin\frac{\pi\alpha}{2}\right] & \alpha\ne 1,\\ \frac{2\beta\gamma}{\pi}\left[1-\gamma_e + \log\frac{1}{\epsilon}\right] & \alpha = 1\end{cases} \qquad (30)
\end{aligned}$$

where $\gamma_e \equiv -\Gamma'(1) \approx 0.577216$ denotes the Euler-Mascheroni constant. The iid summands $\{u_j\}$ for the truncated process will have pdf

$$u_j \sim \nu_\epsilon(du)/\nu_\epsilon^+ = \epsilon^\alpha\left\{\tfrac{1+\beta\,\mathrm{sgn}\,u}{2}\right\}\alpha|u|^{-\alpha-1}\mathbf{1}_{\{|u|>\epsilon\}}\,du,$$

i.e., $\{u_j = r_j\sigma_j\}$ with iid Pareto magnitudes $\{r_j\} \overset{\text{iid}}{\sim} \mathsf{Pa}(\alpha,\epsilon)$ and random signs $\{\sigma_j = \pm 1\}$ with probabilities $(1\pm\beta)/2$. For $0\le t\le T$ we have

$$X_t^\epsilon := \sum_{j\le J_\epsilon}\{u_j:\ 0<s_j\le t\} + t\,(\delta - \mu_\epsilon). \qquad (29b)$$

The truncation error $M_t = [X_t - X_t^\epsilon]$ is an L_2 martingale with predictable quadratic variation $\mathsf{E} M_t^2 = \langle M \rangle_t = t\Sigma_\epsilon$ for real-valued

$$\begin{aligned}\Sigma_\epsilon &\equiv \int_{B_\epsilon} uu' \, \nu(du) \\ &= \frac{2\alpha\gamma\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi} \int_0^\epsilon u^{1-\alpha}\, du \\ &= \frac{2\alpha\gamma\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi(2-\alpha)}\, \epsilon^{2-\alpha},\end{aligned}$$

so by (34c),

$$\mathsf{E}\left[\sup_{0<s\le t} |X_s - X_s^\epsilon|^2\right] \le 4\, t\, \Sigma_\epsilon.$$

2. Cauchy

For the standard **Cauchy** process $X_t \sim \mathsf{St}_\mathsf{M}(1, 0, t, 0)$, $\beta = 0$ and hence $\mu_\epsilon = 0$ (see Eqn (30)), while $\nu_\epsilon^+ = \nu\big(B_\epsilon^c\big) = 2/\pi\epsilon$. Thus the truncation algorithm construction on $[0, T]$ for $\epsilon > 0$ is:

1. Draw $J_\epsilon \sim \mathsf{Po}\big(2T/\pi\epsilon\big)$;
2. Draw J_ϵ iid variates $\{s_j\} \overset{\text{iid}}{\sim} \mathsf{Un}(0, T)$, $\{r_j\} \overset{\text{iid}}{\sim} \mathsf{Pa}(\epsilon, 1)$, and Bernoulli $\{\sigma_j\} = \pm 1$ with probability $\frac{1}{2}$ each;
3. Set
$$X_t^\epsilon := \sum_{j \le J_\epsilon} \{\sigma_j r_j :\ 0 < s_j \le t\}$$
for $0 \le t \le T$.

The predictable variation of the truncation error M_t is $t\Sigma_\epsilon$ for $\Sigma_\epsilon = \int_{|u|\le\epsilon} u^2\, \nu(du) = 2\epsilon/\pi$, so the L_2 error bound of (34c) is

$$\mathsf{E}\left[\sup_{0<s\le T} |X_s - X_s^\epsilon|^2\right] \le \frac{8\ T\,\epsilon}{\pi}$$

and by Markov's inequality, for any $\lambda > 0$,

$$\mathsf{P}\left\{\left[\sup_{0<s\le T} |X_s - X_s^\epsilon|\right] \ge \lambda\right\} \le \frac{8\ T\,\epsilon}{\pi\,\lambda^2}.$$

For $T = 10$, for example, taking $\epsilon \le 1.96 \cdot 10^{-5}$ will ensure that the error bound $|X_s - X_s^\epsilon| \le 0.10$ holds uniformly for $0 \le s \le 10$ with probability 95%, requiring on average $J_\epsilon \approx T\nu_\epsilon^+ \approx 324{,}000$ terms in the summation.

3. Half-Cauchy

For the fully-skewed process $X_t \sim \mathsf{St_M}(1,1,t,0)$, the offset of Eqn (30) is

$$\mu_\epsilon = \frac{2}{\pi}\left[1 - \gamma_e + \log\frac{1}{\epsilon}\right]$$

while again $\Sigma_\epsilon = 2\epsilon/\pi$, so all the $\sigma_j \equiv 1$ and

$$X_t^\epsilon := \sum_{j \le J_\epsilon} \{r_j : \ 0 < s_j \le t\} - t\,\mu_\epsilon$$

for $J_\epsilon \sim \mathsf{Po}\big(\nu_\epsilon^+\big)$ with $\nu_\epsilon^+ = 2/\pi\epsilon$ and $\{r_j\} \overset{\text{iid}}{\sim} \mathsf{Pa}(\alpha,\epsilon)$. As before the martingale maximal bounds are

$$\mathsf{E}\left[\sup_{0<s\le T} |X_s - X_s^\epsilon|^2\right] \le \frac{8\,T\,\epsilon}{\pi} \qquad \mathsf{P}\left\{\sup_{0<s\le T} |X_s - X_s^\epsilon| \ge \lambda\right\} \le \frac{8\,T\,\epsilon}{\pi\,\lambda^2}.$$

4. Gamma

The function $f(u) = u$ is integrable for the Lévy measure $\nu(du) = \alpha u^{-1} e^{-\beta u}\mathbf{1}_{\{u>0\}}$ of the Gamma process $X_t \sim \mathsf{Ga}(\alpha t, \beta)$ at both zero and infinity, so we have a variety of choices about compensation. We can choose $h = \sin$ as we did for the α-stable, or $h = h_\epsilon$; or, ignoring for the moment our usual rules for compensators (boundedness and $h(u) = u + O(|u|^2)$), we can omit compensation entirely (*i.e.*, take $h(u) \equiv 0$) or we can *fully* compensate (so $h(u) \equiv u$). Each leads to the identical algorithm, except for the value of

$$\mu_\epsilon = \int \big[h(u) - u\mathbf{1}_{B_\epsilon}\big]\ \nu(du) = \begin{cases} \alpha\,\mathrm{atan}(1/\beta) - \frac{\alpha}{\beta}\big(1 - e^{-\beta\epsilon}\big) & h(u) = \sin u \\ -\frac{\alpha}{\beta}\big(1 - e^{-\beta\epsilon}\big) & h(u) \equiv 0 \\ (\alpha/\beta)e^{-\beta\epsilon} & h(u) = u \\ 0 & h(u) = h_\epsilon(u) \end{cases}$$

with error bounds arising from $\langle X - X^\epsilon\rangle_t = (\alpha t/\beta^2)\gamma(\epsilon, \beta\epsilon) \le \alpha t \epsilon^2/2$ (Abramowitz and Stegun 1964, §6.5.2), leading to much tighter bounds than for α-stables:

$$\mathsf{E}\left[\sup_{0<s\le T} |X_s - X_s^\epsilon|^2\right] \le 2\alpha T\epsilon^2 \qquad \mathsf{P}\left\{\sup_{0<s\le T} |X_s - X_s^\epsilon| \ge \lambda\right\} \le \frac{2\alpha T\epsilon^2}{\lambda^2},$$

requiring $\epsilon = 0.005$ and only $J_\epsilon \approx T\nu_\epsilon^+ = \alpha T \mathrm{E}_1(\beta\epsilon) \approx 47$ terms to ensure $|X_t - X_t^\epsilon| \le 0.10$ for $0 \le t \le 10$ with probability 95% for $\alpha = \beta = 1$, while just $\nu_\epsilon^+ = 170$ terms will give 99% chance of an error smaller than 10^{-6}.

3.6 Multivariate α-Stable Processes

Let $\Lambda(d\sigma)$ be a finite positive measure on the unit sphere S^{d-1} in $\mathbb{R}^d$, let $0 < \alpha < 2$, and fix $\delta \in \mathbb{R}^d$. An $\mathbb{R}^d$-value random variable X has the **multivariate** α-stable $X \sim \mathsf{St}_{\mathsf{M}}(\alpha, \Lambda, \delta)$ distribution if its log ch.f. $\log \mathsf{E}[e^{i\omega' X}]$ is of the form:

$$i\omega'\delta - \int_{S^{d-1}} \left\{ |\omega'\sigma|^\alpha \left[1 - i\ \tan\tfrac{\pi\alpha}{2} \operatorname{sgn}\omega'\sigma\right] + i\,\omega'\sigma \tan\tfrac{\pi\alpha}{2} \right\} \Lambda(d\sigma) \tag{31a}$$

for $\alpha \neq 1$ or, for $\alpha = 1$,

$$i\omega'\delta - \int_{S^{d-1}} \left\{ |\omega'\sigma| + i\,\frac{2}{\pi}\,\omega'\sigma\ \log|\omega'\sigma| \right\} \Lambda(d\sigma). \tag{31b}$$

Each of these has Lévy-Khinchine form

$$\mathsf{E}[e^{i\omega' X}] = i\omega'\delta + \int_{S^{d-1}\times\mathbb{R}_+} \left\{ e^{i\omega'\sigma r} - 1 - i\,\omega'\sigma \sin r \right\} 2c_\alpha \alpha r^{-\alpha-1}\, dr\, \Lambda(d\sigma)$$

with product Lévy measure $\nu(dr\, d\sigma) = 2c_\alpha \alpha r^{-\alpha-1}\, dr\, \Lambda(d\sigma)$, for the same constant $c_\alpha \equiv \frac{1}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}$ as in Eqn (20). For $0 < \alpha < 1$ it is possible to dispense with compensation, removing the red terms in the displayed equations, leading to the $\mathsf{St}_{\mathsf{A}}(\alpha, \Lambda, \delta)$ parametrization.

The measure Λ determines both the intensity $\gamma = \Lambda(S^{d-1})$ of the process and its degree of asymmetry $\beta = \int_{S^{d-1}} \sigma\Lambda(d\sigma)/\gamma$ (and in $d = 1$ dimension would coincide with the usual $\mathsf{St}_{\mathsf{M}}(\alpha, \beta, \gamma, \delta)$ distribution). In $d \geq 2$ dimensions the α-stable family of distributions is much richer than in $d = 1$ dimension, though, because it entails an arbitrary angular distribution on S^{d-1} while probability distributions on the two-point $S^0 = \{\pm 1\}$ are determined by a single number $\beta = \int_{S^0} \sigma\Lambda(d\sigma) = (p_+ - p_-) \in [-1, 1]$.

One specific example is the **Independent** α**-Stables** $X = (X_1, ..., X_d)$ with $X_j \overset{\text{ind}}{\sim} \mathsf{St}_{\mathsf{M}}(\alpha, \beta_j, \gamma_j, \delta_j)$, all with the same shape parameter α. In this case the vector $X \sim \mathsf{St}_{\mathsf{M}}(\alpha, \Lambda, \delta)$ with Λ the sum of $2d$ point masses of magnitude $\gamma_j(1 \pm \beta_j)/2$ at $\pm e_j$, the unit vector in the j'th coordinate direction, and $\delta = (\delta_1, \ldots, \delta_d)$.

A second specific example is the d-dimensional **Symmetric** α**-Stable** distribution $X \sim \mathsf{St}_{\mathsf{M}}(\alpha, \Lambda, 0)$ with $\Lambda(d\sigma)$ proportional to the uniform distribution on S^{d-1}. In this case for any $\omega \in \mathbb{R}^d$ the inner product $\omega' X \sim \mathsf{St}_{\mathsf{M}}(\alpha, 0, \gamma^*, 0)$ has a univariate SαS distribution with

$$\gamma^* = |\omega|^\alpha \int_{S^{d-1}} |\sigma_1|^\alpha \Lambda(d\sigma) = |\omega|^\alpha \frac{\Gamma(\frac{\alpha+1}{2})\Gamma(\frac{d}{2})}{\Gamma(\frac{\alpha+d}{2})\sqrt{\pi}}\, \Lambda(S^{d-1}).$$

3.6.1 Generating the Multivariate α-Stable

To generate an SII process $X_t \sim \mathsf{St_M}(\alpha, \Lambda t, \delta)$, begin by computing for $r > 0$

$$\begin{aligned}
\nu_r^+ &= \nu\big(B_r^c\big) = \Lambda(S^{d-1}) \int_r^\infty 2c_\alpha \alpha x^{-\alpha-1}\, dx \\
&= \frac{2\gamma}{\pi}\Gamma(\alpha)\sin\tfrac{\pi\alpha}{2} r^{-\alpha} && \text{(32a)} \\
\mu_\epsilon &= \int_{\mathbb{R}^d} \left[h(u) - h_\epsilon(u)\right]\, \nu(du) \\
&= \int_{S^{d-1}\times\mathbb{R}_+} \left[\sigma \sin u - \sigma u \mathbf{1}_{\{r<\epsilon\}}\right]\, 2c_\alpha \alpha r^{-\alpha-1} dr\, \Lambda(d\sigma) \\
&= \gamma\beta \int_{\mathbb{R}_+} \left[\sin u - u\mathbf{1}_{\{r<\epsilon\}}\right]\, 2c_\alpha \alpha r^{-\alpha-1} dr \\
&= \frac{2\gamma\beta}{\pi} \begin{cases} \left[\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\frac{\epsilon^{1-\alpha}}{\alpha-1} - \tan\frac{\pi\alpha}{2}\right] & \alpha \neq 1 \\ \left[1 - \gamma_e + \log\frac{1}{\epsilon}\right] & \alpha = 1. \end{cases} && \text{(32b)}
\end{aligned}$$

For $\alpha < 1$ it is possible to skip compensation; in that case, omit the "$\tan\frac{\pi\alpha}{2}$" term in (32b). Now the Truncation Algorithm steps are:

1. Fix $\epsilon > 0$ and $T > 0$, and draw $J_\epsilon \sim \mathsf{Po}\big(T\nu_\epsilon^+\big)$;
2. Draw J_ϵ iid variates $\{s_j\} \overset{\text{iid}}{\sim} \mathsf{Un}(0,T)$, $\{r_j\} \overset{\text{iid}}{\sim} \mathsf{Pa}(\alpha,\epsilon)$, and $\{\sigma_j\} \overset{\text{iid}}{\sim} \Lambda(d\sigma)/\gamma$;
3. For $0 \le t \le T$, set

$$X_t^\epsilon := \sum_{j \le J_\epsilon} \{r_j \sigma_j :\ 0 < s_j \le t\} + t\,(\delta - \mu_\epsilon). \tag{33}$$

Just as in one dimension, this is tolerably efficient for $0 < \alpha < 1$ but the number of terms required for an adequate approximation becomes excessive for $\alpha \ge 1$, with martingale-based error estimates readily available.

The ILM algorithm is similar; since ν is a product measure, the magnitudes $\{r_n\}$ and directions $\{\sigma_n\}$ are independent for both algorithms.

3.7 Multivariate Extreme Value Theory

Multivariate α-stables are a great source of explicit examples.

4 Numerical Methods

4.1 Prior and Posterior distributions

4.2 Gamma/Poisson and Gibbs

4.3 Reversible Jump MCMC

5 Applications

5.1 Biodiversity

Estimating the latent mean intensity (in trees/m^2) for eight species of trees in Duke Forest (Wolpert and Ickstadt 1998a).

5.2 Proteomics

Identifying and quantifying proteins in serum samples based on the time-of-flight of charged molecular fragments in an electric field (Clyde et al. 2006; House et al. 2011).

5.3 Nitrate Concentrations

Estimating the unmeasured concentrations of nitrates at locations in the eastern US, based on irregularly-spaced measurements (Woodard et al. 2010).

5.4 Epidemiology

Exploring the etiology of "severe wheeze" (childhood asthma), by modeling excess disease rate (cases per 100 at-risk individuals per year) after adjusting for known risk factors as a latent (and possibly zero) function of space (Best et al. 2000a,b, 2002).

5.5 Gamma Ray Bursts

Go see Mary Beth's 395 presentation on 2012-03-12, 4:25pm, 116 Old Chem Building.

6 Stationary ID Processes

6.1 MISTI

Characterizing all integer-valued time-reversible stationary Markov processes with infinitely-divisible distributions (Wolpert and Brown 2011).

6.2 Six $\mathsf{AR}(1)$-like Gamma Processes

Count 'em. All with identical univariate marginals and auto-correlation functions, but all different. Some are ID, some are Markov, some are time-reversible, one is a diffusion.

6.3 α-Stable $\mathsf{AR}(1)$ Processes

A Appendix: Martingales

Let $\{\mathcal{F}_t\} \subset \mathcal{F}$ be an increasing family of sub-σ-algebras on a probability space $(\Omega, \mathcal{F}, \mathsf{P})$, indexed by $t \in \mathcal{T}$ for an ordered set $\mathcal{T}$ (usually all or a subset of $\mathbb{R}$ or $\mathbb{Z}$). such a family is called a *filtration*. A *martingale* is a family $\{M_t\} \subset L_1(\Omega, \mathcal{F}, \mathsf{P})$ of integrable random variables (also indexed by $t \in \mathcal{T}$) taking values in some vector space E (usually $\mathbb{R}$ or $\mathbb{R}^d$) and for all $s < t \in \mathcal{T}$ satisfying the property

$$M_s = \mathsf{E}[M_t \mid \mathcal{F}_s].$$

In particular this implies M_t is $\mathcal{F}_t$-measurable for all $t \in \mathcal{T}$; if the filtration isn't specified, then implicitly it is taken to be $\sigma\{M_s : s \le t\}$, the smallest σ-algebra with this property.

The *quadratic variation* of a square-integrable $\mathbb{R}^d$-valued martingale M_t is the $d \times d$-matrix-valued process given by the limit

$$[M]_t \equiv \lim \sum_{0 \le i < n} (M_{t_{i+1}} - M_{t_i})\,(M_{t_{i+1}} - M_{t_i})'$$

where the limit is taken over sequences $0 = t_0 < t_1 < \cdots < t_n = t$ as $n \to \infty$ with $\max |t_{i+1} - t_i| \to 0$. For two L_2 martingales M^1 and M^2, the cross-variation is

$$\begin{aligned}[M^1, M^2]_t &\equiv \lim \sum_{0 \le i < n} (M^1_{t_{i+1}} - M^1_{t_i})\,(M^2_{t_{i+1}} - M^2_{t_i})' \\ &= \frac{1}{4}\Big([M^1 + M^2]_t - [M^1 - M^2]_t\Big).\end{aligned}$$

The quadratic variation $[M]_t$ for most martingales M_t associated with Lévy processes are random processes with jumps. The "previsible projection" of $[M]_t$ can be expressed

$$\langle M\rangle_t \equiv \lim \sum_{0\le i<n} \mathsf{E}\big[(M_{t_{i+1}} - M_{t_i})\,(M_{t_{i+1}} - M_{t_i})' \,\big|\, \mathcal{F}_{t_i}\big],$$

often a deterministic function for Lévy-related martingales. Writing

$$M_t = M_0 + \sum_{0\le i<n} (M_{t_{i+1}} - M_{t_i})$$

as a telescoping sum and squaring, we find

$$\mathsf{E}M_t^2 = \mathsf{E}M_0^2 + \mathsf{E}[M]_t = \mathsf{E}M_0^2 + \mathsf{E}\langle M\rangle_t$$

for all t. In the applications in these notes, $M_0 \equiv 0$ and $\langle M\rangle_t$ is non-random, making this a convenient way to calculate $\mathsf{E}M_t^2 = \langle M\rangle_t$.

For example, if $X_t \sim \mathsf{Po}(\lambda t)$ is an SII Poisson process with rate $\lambda > 0$, then $M_t \equiv [X_t - \lambda t]$ is an L_2 martingale, with

$$[M]_t = X_t \qquad \text{and} \qquad \langle M\rangle_t = \lambda t$$

and

$$\mathsf{E}(X_t - \lambda t)^2 = \mathsf{E}X_t = \lambda t.$$

Doob's Maximal Inequalities

Joseph Doob (1990, Ch. VII) proved a number of exceptionally useful bounds for the *maxima* over time intervals of real-valued martingales. Let M denote a real-valued martingale, and define

$$M_t^* = \sup_{0\le s\le t} M_s$$
$$|M|_t^* = \sup_{0\le s\le t} |M_s|$$

For $\lambda > 0$ and $p > 1$, Doob's inequalities include:

$$\mathsf{P}\big[M_t^* \ge \lambda\big] \le \frac{\mathsf{E}M_t^+}{\lambda} \tag{34a}$$

$$\mathsf{P}\big[|M|_t^* \ge \lambda\big] \le \frac{\mathsf{E}|M_t|}{\lambda} \tag{34b}$$

$$\mathsf{E}\big[(|M|_t^*)^p\big] \le \left[\frac{p}{p-1}\right]^p \,\mathsf{E}|M_t|^p \tag{34c}$$

where $M^+ \equiv (0 \vee M)$. Equation (34b) is a strong extension of Markov's inequality, which would have asserted the same bound merely for $\mathsf{P}\big[|M_t| \geq \lambda\big]$. For $p=2$, Eqn (34c) bounds the L_2 norm of the sample path maximum $|M|_t^*$ by twice that of M_t, the process at a single point.

Here's a sketch of the proofs. Fix $\lambda > 0$ and a martingale M. Define the *stopping time* $\tau \equiv \inf\{t \geq 0 :\ M_t \geq \lambda\}$ (or infinity if M_t never exceeds λ). By the optional sampling theorem, $M_{t\wedge\tau}$ is also a martingale, and for all $t \geq 0$, $\mathsf{E}M_t = \mathsf{E}M_{t\wedge\tau} = \mathsf{E}M_0$, so

$$\begin{aligned}
\mathsf{E}[M_t] = \mathsf{E}[M_{t\wedge\tau}] &= \mathsf{E}[M_\tau \mathbf{1}_{\{\tau\leq t\}}] + \mathsf{E}[M_t \mathbf{1}_{\{\tau>t\}}] \\
&\geq \mathsf{E}[\lambda \mathbf{1}_{\{\tau\leq t\}}] + \mathsf{E}[M_t \mathbf{1}_{\{\tau>t\}}] \\
&= \lambda \mathsf{P}[\tau \leq t] + \mathsf{E}[M_t] - \mathsf{E}[M_t \mathbf{1}_{\{\tau\leq t\}}] \\
\lambda \mathsf{P}[\tau \leq t] \leq \mathsf{E}[M_t \mathbf{1}_{\{\tau\leq t\}}] &\leq \mathsf{E}[M_t^+ \mathbf{1}_{\{\tau\leq t\}}] \leq \mathsf{E}[M_t^+],
\end{aligned}$$

proving Eqn (34a) since the event $[\tau \leq t]$ is the same as $[M_t^* \geq \lambda]$.

Since $[-M_t]$ is also a martingale and $[-M_t]^+ = [0 \vee (-M_t)] = M_t^-$,

$$\begin{aligned}
\mathsf{P}[\sup_{0\leq s\leq t} |M_s| \geq \lambda] &\leq \mathsf{P}[\sup_{0\leq s\leq t} M_s \geq \lambda] + \mathsf{P}[\inf_{0\leq s\leq t} M_s \leq -\lambda] \\
&\leq \frac{\mathsf{E}M_t^+}{\lambda} + \frac{\mathsf{E}M_t^-}{\lambda} = \frac{\mathsf{E}|M_t|}{\lambda},
\end{aligned}$$

proving Eqn (34b). In fact we proved something slightly stronger, that we'll need below:

$$\mathsf{P}\left[|M|_t^* \geq \lambda\right] \leq \frac{1}{\lambda}\mathsf{E}\left[|M_t| \mathbf{1}_{\{|M|_t^*\geq\lambda\}}\right].$$

Finally, by Fubini's theorem, the nonnegative random variable $Y \equiv |M|_t^*$ satisfies

$$\begin{aligned}
\mathsf{E}[Y^p] &= \int_0^\infty p\lambda^{p-1}\mathsf{P}[Y \geq \lambda]\,d\lambda \leq \int_0^\infty p\lambda^{p-1}\frac{1}{\lambda}\mathsf{E}\left[|M_t| \mathbf{1}_{\{Y\geq\lambda\}}\right]\,d\lambda \\
&= \mathsf{E}\,|M_t| \int_0^Y p\lambda^{p-2}\,d\lambda \qquad = \frac{p}{p-1}\mathsf{E}\left[|M_t|\,Y^{p-1}\right].
\end{aligned}$$

Hölder's inequality asserts that $\mathsf{E}[A\,B] \leq \mathsf{E}[A^p]^{(1/p)}\,\mathsf{E}[B^q]^{(1/q)}$ for any positive random variables A and B, for conjugate exponent $q = \frac{p}{p-1}$. Applying this with $A = |M_t|$ and $B = Y^{p-1}$, and noting $(p-1)q = p$ and $1 - \frac{1}{q} = \frac{1}{p}$,

$$\begin{aligned}
\mathsf{E}[Y^p] &\leq \frac{p}{p-1}\left[\mathsf{E}|M_t|^p\right]^{1/p}\left[\mathsf{E}Y^p\right]^{1/q} \\
\mathsf{E}[Y^p]^{1/p} &\leq \frac{p}{p-1}\left[\mathsf{E}|M_t|^p\right]^{1/p},
\end{aligned}$$

proving Eqn (34c).

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