Robert's α-Stable Notes

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1 Log Ch.F. for Stable Distributions

General $\mathsf{St}(\alpha,\beta,\gamma,\delta)$**:**

$$\begin{aligned}\log\phi_X(\omega) &= \begin{cases} i\delta\omega-\gamma|\omega|^\alpha+i\beta\gamma\tan\frac{\pi\alpha}{2}\{|\omega|^\alpha\operatorname{sgn}\omega-\omega\} & \alpha\neq 1\\ i\delta\omega-\gamma|\omega| \quad -\frac{2}{\pi}\,i\beta\gamma\omega\log|\omega| & \alpha=1\end{cases}\\ &= \begin{cases} i\omega\left(\delta-\beta\gamma\tan\frac{\pi\alpha}{2}\right)-\gamma|\omega|^\alpha\left(1-i\beta\tan\frac{\pi\alpha}{2}\operatorname{sgn}\omega\right) & \alpha\neq 1\\ i\omega\left(\delta-2\beta\gamma\log|\omega|/\pi\right)-\gamma|\omega| & \alpha=1\end{cases}\end{aligned}$$

$0<\alpha\le 2$, $-1\le\beta\le 1$, $0<\gamma<\infty$, $-\infty<\delta<\infty$.

Standard $\mathsf{St}(\alpha,\beta,1,0)$**:**

$$\log\phi_X(\omega)=\begin{cases}-|\omega|^\alpha+i\beta\tan\frac{\pi\alpha}{2}\{|\omega|^\alpha\operatorname{sgn}\omega-\omega\} & \alpha\neq 1\\ -|\omega| \quad -\frac{2}{\pi}\,i\beta\omega\log|\omega| & \alpha=1\end{cases}$$

Symmetric $\mathsf{St}(\alpha,0,\gamma,0)$**:**

$$\log\phi_X(\omega)=-\gamma|\omega|^\alpha$$

Skewed $\mathsf{St}(\alpha,1,\gamma,0)$**:**

$$\log\phi_X(\omega)=\begin{cases}-\gamma|\omega|^\alpha+i\gamma\tan\frac{\pi\alpha}{2}\{|\omega|^\alpha\operatorname{sgn}\omega-\omega\} & \alpha\neq 1\\ -\gamma|\omega| \quad -\frac{2}{\pi}\,i\gamma\omega\log|\omega| & \alpha=1\end{cases}$$

Standard Stable Lévy Measure

We now turn to computing the Lévy measure for the standard $\mathsf{St}(\alpha, \beta, 1, 0)$ stable distribution. For $\alpha < 0$ and $\lambda > 0$ set $\nu_+(u) := u^{-\alpha-1}\, e^{-u\lambda}$ on $\mathbb{R}_+$ and compute

$$
\begin{aligned}
\int_0^\infty e^{i\omega u} \nu_+(du) &= \int_0^\infty e^{i\omega u} u^{-\alpha-1} e^{-u\lambda} du \\
&= \int_0^\infty u^{-\alpha-1} e^{-u(\lambda - i\omega)} du \\
&= \Gamma(-\alpha)(\lambda - i\omega)^\alpha \\
&= \Gamma(-\alpha)(\lambda^2 + \omega^2)^{\alpha/2} e^{-i\alpha \operatorname{atan} \frac{\omega}{\lambda}} \\
\int_0^\infty \nu_+(du) &= \Gamma(-\alpha)\lambda^\alpha \\
\int_0^\infty \sin u\, \nu_+(du) &= \Im \left[\int_0^\infty u^{-\alpha-1} e^{-u(\lambda - i)}\, du \right] \\
&= \Gamma(-\alpha) \Im (\lambda - i)^\alpha \\
&= -\Gamma(-\alpha)(\lambda^2 + 1)^{\alpha/2} \sin(\alpha \operatorname{atan} \tfrac{1}{\lambda}).
\end{aligned}
$$

Thus the ch.f. with Lévy measure given on $\mathbb{R}_+$ by $\nu_+(du) := u^{-\alpha-1} e^{-u\lambda}\, du$ is

$$
\begin{aligned}
\log \phi_+(\omega) &= \int_0^\infty \left(e^{i\omega u} - 1 - i\omega \sin u \right) \nu_+(du) \\
&= \Gamma(-\alpha) \Big\{ (\lambda - i\omega)^\alpha - \lambda^\alpha - i\omega \tfrac{1}{2} [(\lambda - i)^\alpha - (\lambda + i)^\alpha] \Big\} \\
&= \Gamma(-\alpha) \Big\{ (\lambda^2 + \omega^2)^{\alpha/2} \big[\cos(\alpha \operatorname{atan} \frac{\omega}{\lambda}) - i \sin(\alpha \operatorname{atan} \frac{\omega}{\lambda}) \big] \\
&\qquad\qquad - \lambda^\alpha + i\omega(\lambda^2 + 1)^{\alpha/2} \sin(\alpha \operatorname{atan} \frac{1}{\lambda}) \Big\}
\end{aligned}
$$

This is analytic in α (away from the positive integers), so we first take the analytic continuation to the segment $0 < \alpha < 2$, then the limit as $\lambda \to 0$:

$$
\begin{aligned}
&\to \Gamma(-\alpha) \left\{ |\omega|^\alpha \cos \frac{\pi\alpha}{2} - i \sin \frac{\pi\alpha}{2} \big(|\omega|^\alpha \operatorname{sgn} \omega - \omega \big) \right\} \quad (\text{as } \lambda \to 0) \\
&= \frac{-\pi}{2\alpha\Gamma(\alpha) \sin \frac{\pi\alpha}{2}} \left\{ |\omega|^\alpha - i \tan \frac{\pi\alpha}{2} \big(|\omega|^\alpha \operatorname{sgn} \omega - \omega \big) \right\}.
\end{aligned}
$$

Similarly the ch.f. with Lévy measure $\nu_-(u) := |u|^{-\alpha-1}e^{-|u|\lambda}\,du$ on $\mathbb{R}_-$ is

$$\begin{aligned}\log\phi_-(\omega) &= \int_{-\infty}^{0}\big(e^{i\omega u}-1-i\omega\sin u\big)\,\nu_-(du)\\ &\to \frac{-\pi}{2\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}\Big\{|\omega|^\alpha + i\tan\frac{\pi\alpha}{2}\big(|\omega|^\alpha\operatorname{sgn}\omega-\omega\big)\Big\}\end{aligned}$$

so for $-1\le\beta\le 1$ the ch.f. for Lévy measure

$$\nu(du) = \frac{\alpha}{\pi}\,\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\,(1+\beta\operatorname{sgn}u)\,|u|^{-\alpha-1}\,du \tag{1}$$

is

$$\begin{aligned}\log\phi(\omega) &= \int_{-\infty}^{\infty}\big(e^{i\omega u}-1-i\omega\sin u\big)\,\nu(du)\\ &= -|\omega|^\alpha + i\beta\tan\frac{\pi\alpha}{2}\big(|\omega|^\alpha\operatorname{sgn}\omega-\omega\big),\end{aligned}$$

exactly the Zolotarev (1986) (M) parametrization of the $\mathsf{St}(\alpha,\beta,1,0)$ distribution, also recommended by Cheng and Liu (1997) and Nolan (1998a). The Lévy measure and ch.f. for case $\alpha=1$ follow from a similar argument or simply from continuity:

$$\nu(du) = \frac{1+\beta\operatorname{sgn}u}{\pi}\;|u|^{-2}\,du \qquad \log\phi(\omega) = -|\omega| - \frac{2\beta i}{\pi}\omega\log|\omega|.$$

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Four-Parameter Stable Distributions

Cheng and Liu (1997) and Nolan (1998a) take the remaining parameters γ and δ of the four-parameter family of stable distributions $X\sim\mathsf{St}(\alpha,\beta,\gamma,\delta)$ to be scale and location parameters, respectively, so $X=\gamma Z+\delta$ for some $Z\sim\mathsf{St}(\alpha,\beta,1,0)$; with this choice the parameters γ^*, δ^* governing the sum $X_+ := \sum X_i \sim \mathsf{St}(\alpha,\beta,\gamma^*,\delta^*)$ of independent stable random variables $X_i\sim\mathsf{St}(\alpha,\beta,\gamma_i,\delta_i)$ is exceedingly awkward, and it is difficult to describe the stable random fields we will need below. We instead take γ to be a rate parameter (as in Zolotarev, 1986) and δ a location parameter, leading to the general form

$$\begin{aligned}
\nu(du) &= \frac{\alpha\gamma}{\pi}\,\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\,(1+\beta\,\operatorname{sgn} u)\,|u|^{-\alpha-1}\,du, &(2)\\
\log\phi(\omega) &= i\delta\omega + \int_{-\infty}^{\infty}\big(e^{i\omega u} - 1 - i\omega\sin u\big)\,\nu(du) &(3)\\
&= i\delta\omega - \frac{\gamma}{2}\sec\frac{\pi\alpha}{2}\Big\{(1+\beta)(-i\omega)^{\alpha} + \beta\omega(+i)^{\alpha}\\
&\qquad\qquad +(1-\beta)(+i\omega)^{\alpha} - \beta\omega(-i)^{\alpha}\Big\}\\
&= \begin{cases} i\delta\omega - \gamma|\omega|^{\alpha} + i\beta\gamma\tan\frac{\pi\alpha}{2}\big(|\omega|^{\alpha}\operatorname{sgn}\omega - \omega\big) & \alpha \neq 1;\\ i\delta\omega - \gamma|\omega| \;\; - i\beta\gamma\frac{2}{\pi}\,\omega\log|\omega| & \alpha = 1. \end{cases} &(4)
\end{aligned}$$

With this choice the sum $X_+ := \sum X_i$ of independent stable random variables $X_i \sim \mathsf{St}(\alpha,\beta_i,\gamma_i,\delta_i)$ has a stable distribution $X_+ \sim \mathsf{St}(\alpha,\bar\beta,\gamma_+,\delta_+)$ with parameters given simply by the sums $\gamma_+ := \sum\gamma_i$, $\delta_+ := \sum\delta_i$ and average $\bar\beta := \sum\beta_i\gamma_i/\gamma_+$.

Five-Parameter Tempered Stable Distributions

Mantegna and Stanley (1994) noticed that partial sums of Pareto random variables truncated at some bound $l < \infty$ would behave much like α-Stable processes initially, but by the CLT would behave much like Gaussian processes at large time; they named them Truncated Lévy flights. Koponen (1995) replaced the hard cut-off with an exponential fall-off to produce what we will call a tempered Pareto distribution, with p.d.f.

$$X \sim c^{-1}\,x^{-\alpha-1}\,e^{-\lambda x}\;\mathbf{1}_{(\epsilon,\infty)}(x) \tag{5}$$

for some parameters $\alpha,\epsilon,\lambda > 0$, where evidently c is given by

$$\begin{aligned}
c &= \lambda^{\alpha}\Gamma(-\alpha,\lambda\epsilon)\\
&= \frac{1}{\alpha}\big[\epsilon^{-\alpha}e^{-\lambda\epsilon} - \lambda^{\alpha}\Gamma(1-\alpha,\lambda\epsilon)\big].
\end{aligned}$$

This is the probability distribution of the jumps exceeding ϵ of the *Tempered α-Stable Process*, a Lévy process with Lévy measure differing only by a factor of $e^{-\lambda|u|}$ from that of the α-Stable process of Eqn. (2),

$$\nu(du) = \frac{\alpha\gamma}{\pi}\,\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\,(1+\beta\,\operatorname{sgn} u)\,|u|^{-\alpha-1}\,e^{-\lambda|u|}\,du, \tag{6}$$

with *uncompensated* log ch.f. given for $0 < \alpha < 1$ and $t = 1$ by

$$\begin{aligned}
\log \phi(\omega) &= i\delta\omega + \int_{-\infty}^{\infty} \left(e^{i\omega u} - 1\right) \nu(du) \\
&= i\delta\omega - \frac{\gamma}{2\cos\frac{\pi\alpha}{2}} \Big\{ (1+\beta)(\lambda - i\omega)^\alpha + (1-\beta)(\lambda + i\omega)^\alpha - 2\lambda^\alpha \Big\} \\
&= i\delta\omega + i\beta\gamma \sec\frac{\pi\alpha}{2} (\omega^2 + \lambda^2)^{\alpha/2} \sin\left(\alpha \operatorname{atan}\frac{\omega}{\lambda}\right) \\
&\qquad - \gamma \sec\frac{\pi\alpha}{2} \Big\{ (\omega^2 + \lambda^2)^{\alpha/2} \cos\left(\alpha \operatorname{atan}\frac{\omega}{\lambda}\right) - \lambda^\alpha \Big\}, \qquad (7)
\end{aligned}$$

which (with $\delta = 0$ and $\beta = 1$) converges as $\alpha \to 0$, $\gamma \to \infty$ with $\alpha\gamma \to c$ to the $\mathsf{Ga}(c, \lambda)$ log ch.f., and converges to the uncompensated α-Stable $\mathsf{St}(\alpha, \beta, \gamma, \delta)$ for $0 < \alpha < 1$ as $\lambda \to 0$. For $\alpha = \frac{1}{2}$, $\beta = 1$, $\gamma = \sqrt{b}$, $\lambda = a/2$ it coincides with the Inverse Gamma $\mathsf{IG}(a, b)$ distribution.

Both the Tempered Pareto and Tempered Stable distributions have moments of all orders; for example, the moments of the Tempered Pareto $\mathsf{TP}(\alpha, \epsilon; \lambda)$ are:

$$\mathsf{E}X^p = \lambda^{-p}\Gamma(p - \alpha, \lambda\epsilon)/\Gamma(-\alpha, \lambda\epsilon)$$

while the Tempered Stable process $X_t \sim \mathsf{TS}(\alpha, \beta, \gamma t, \delta t; \lambda)$ has mean and variance:

$$\begin{aligned}
\mathsf{E}[X_t] &= \delta t + \alpha\beta\gamma\lambda^{\alpha-1} \sec(\frac{\pi\alpha}{2}) t \\
\mathsf{V}[X_t] &= \alpha(1-\alpha)\gamma\lambda^{\alpha-2} \sec(\frac{\pi\alpha}{2}) t
\end{aligned}$$

for $0 < \alpha < 1$, $-1 \le \beta \le 1$, $\gamma > 0$, $\delta \in \mathbb{R}$, $\lambda > 0$.

Both fully-compensated (with $h(u) = u$) and partially compensated (with, for example, $h(u) = \sin u$) versions can be constructed for all $0 < \alpha < 2$, leading to an additional imaginary term proportional to $i\omega\beta\gamma$ in Eqn. (7); for the fully-compensated case, for example, and $t = 1$,

$$\begin{aligned}
\log \phi(\omega) &= i\omega\Big[\delta - \beta\gamma\alpha\lambda^{\alpha-1} \sec(\frac{\pi\alpha}{2})\Big] \\
&\qquad - \frac{\gamma}{\cos\frac{\pi\alpha}{2}} \Big\{ (\omega^2 + \lambda^2)^{\alpha/2} \big[\cos\left(\alpha \operatorname{atan}\frac{\omega}{\lambda}\right) - i\beta \sin\left(\alpha \operatorname{atan}\frac{\omega}{\lambda}\right) \big] - \lambda^\alpha \Big\}
\end{aligned}$$

leading to a process with $\mathsf{E}[X_t] = \delta t$, and the same variance as before, but now well-defined for $0 < \alpha < 2$. The conventional compensator $h(u) = \sin u$ leads to a similar expression,

$$\begin{aligned}\log\phi(\omega) = i\omega\Big[\delta - \beta\gamma\sec\frac{\pi\alpha}{2}\big(\lambda^2+1\big)^{\alpha/2}\sin(\alpha\,\mathrm{atan}\,\frac{1}{\lambda})\Big]\\ -\gamma\sec\frac{\pi\alpha}{2}\Big\{(\omega^2+\lambda^2)^{\alpha/2}\big[\cos\big(\alpha\,\mathrm{atan}\,\frac{\omega}{\lambda}\big) - i\beta\sin\big(\alpha\,\mathrm{atan}\,\frac{\omega}{\lambda}\big)\big] - \lambda^\alpha\Big\}\end{aligned}$$

or, in the limit as $\alpha \to 1$,

$$\begin{aligned}= i\omega\delta + i\omega\ \frac{2\gamma\beta}{\pi}\left\{\frac{1}{2}\log\frac{\lambda^2+1}{\lambda^2+\omega^2} - \frac{\lambda}{\omega}\,\mathrm{atan}\,\frac{\omega}{\lambda}\right\}\\ -\frac{2\gamma}{\pi}\left\{\frac{\lambda}{2}\log\frac{\lambda^2}{\lambda^2+\omega^2} + \omega\ \mathrm{atan}\,\frac{\omega}{\lambda}\right\}.\end{aligned}$$

In the limit as $\lambda \to 0$ these converge to the usual α-Stable log chf,

$$\to \begin{cases} -\gamma|\omega|^\alpha + i\omega\Big\{\delta + \beta\gamma\tan\frac{\pi\alpha}{2}(|\omega|^{\alpha-1}-1)\Big\} & \alpha \neq 1\\ -\gamma|\omega| \ \ + i\omega\Big\{\delta - \beta\gamma\frac{2}{\pi}\log|\omega|\Big\} & \alpha = 1\end{cases}$$

Another Tempered Distribution

Sample paths of the SII process with $X_t \sim \mathsf{St}(\alpha,\beta,\gamma t,\delta,\lambda)$ of the previous section exhibit countably-many jumps in any open time interval. For any $\epsilon > 0$, the number N_ϵ of positive jumps $\Delta X_t \equiv [X_t - X_{t-}] > \epsilon$ exceeding ϵ during an interval $(t_1,\ t_2]$ has a Poisson distribution with mean

$$\mathsf{E}N_\epsilon = (t_2 - t_1)\frac{\alpha\gamma}{\pi}(1+\beta)\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\lambda^\alpha\Gamma(-\alpha;\lambda\epsilon) \tag{8a}$$

and, conditional on N_ϵ, the jumps themselves are independent with identical Tempered Pareto distributions with probability density function

$$f(u) = \frac{\lambda^{-\alpha}}{\Gamma(-\alpha;\lambda\epsilon)}\,u^{-\alpha-1}e^{-\lambda u}\mathbf{1}_{\{u>\epsilon\}}. \tag{8b}$$

Because the likelihood arising from Eqn. (8) is so awkward (making both Bayesian and sampling-theory inference challenging for observations of $\{\Delta X_t\}$), an attractive alternative with very similar properties is the Tempered Stable with Lévy measure

$$\nu(du) = \frac{\gamma}{\pi}\,\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\,(1+\beta\operatorname{sgn}u)\,(\alpha+\lambda|u|)\,|u|^{-\alpha-1}\,e^{-\lambda|u|}\,du \tag{9}$$

(cf. Eqn. (6)). For the fully-skewed case $\beta = 1$, the resulting log ch. f. is

$$\begin{aligned}\log\phi(\omega) = \gamma\sec\frac{\pi\alpha}{2}\omega\big(\lambda^2+\omega^2\big)^{\frac{\alpha-1}{2}}\sin\Big((\alpha-1)\operatorname{atan}\frac{\omega}{\lambda}\Big)\\ + i\gamma\omega\sec\frac{\pi\alpha}{2}\Big\{\big(\lambda^2+\omega^2\big)^{\frac{\alpha-1}{2}}\cos\Big((\alpha-1)\operatorname{atan}\frac{\omega}{\lambda}\Big)\\ - \big(\lambda^2+1\big)^{\frac{\alpha-1}{2}}\cos\Big((\alpha-1)\operatorname{atan}\frac{1}{\lambda}\Big)\Big\}\end{aligned}$$

but now the number, density and CDF for the jumps exceeding ϵ have the simpler form

$$\mathsf{E}N_\epsilon = (t_2-t_1)\frac{\alpha\gamma}{\pi}(1+\beta)\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\epsilon^{-\alpha}e^{-\lambda\epsilon} \tag{10a}$$

$$\begin{aligned}f(u) &= \epsilon^{\alpha}(\alpha+\lambda u)u^{-\alpha-1}e^{-\lambda(u-\epsilon)}\mathbf{1}_{\{u>\epsilon\}}, \qquad (10b)\\ F(u) &= 1-(u/\epsilon)^{-\alpha}e^{-\lambda(u-\epsilon)}, \qquad u>\epsilon\end{aligned}$$

making likelihood-based inference practical even for censored data.

Location-Scale Family

If X is a stable random variable $X \sim \mathsf{St}(\alpha,\beta,\gamma,\delta)$ and $Y := \eta+\sigma X$ is in the same location-scale family for some $\eta,\sigma\in\mathbb{R}$ (note σ need not be positive) then $Y \sim \mathsf{St}(\alpha^*,\beta^*,\gamma^*,\delta^*)$ with

$$\begin{aligned}\alpha^* &= \alpha\\ \beta^* &= \beta\operatorname{sgn}\sigma\\ \gamma^* &= \gamma|\sigma|^{\alpha}\\ \delta^* &= \begin{cases}\eta+\sigma\delta+\beta\gamma\tan\frac{\pi\alpha}{2}\,\sigma(|\sigma|^{\alpha-1}-1) & \alpha\neq 1\\ \eta+\sigma\delta-\beta\gamma\frac{2}{\pi}\,\sigma\log|\sigma| & \alpha=1\end{cases}\end{aligned}$$

It is straightforward to solve these for the parameters η and σ needed to achieve standard form $Y := \eta+\sigma X \sim \mathsf{St}(\alpha,\beta,1,0)$ with $\gamma^*=1$, $\delta^*=0$:

$$\sigma = \gamma^{-1/\alpha} \qquad\qquad \eta = \begin{cases}-\sigma\delta+\beta\tan\frac{\pi\alpha}{2}(\sigma^{1-\alpha}-1) & \alpha\neq 1\\ -\sigma\delta+\beta\frac{2}{\pi}\,\log\sigma & \alpha=1.\end{cases}$$

Arbitrary Compensators

The Lévy-Khinchine representation

$$\log \phi(\omega) = i\omega\delta^* + \int_{\mathbb{R}} \left[e^{i\omega u} - 1 - i\omega h(u)\right] \nu(du)$$

may be written with any bounded Borel compensator $h(u) = u + O(u^2)$ near $u \approx 0$. The choice $h(u) = \sin u$ leads to $\delta^* = \delta$, as in Eqn. (3), while any other bounded compensator $h(u) = u + O(u^2)$ may require a different shift,

$$\delta^* = \delta + \int [h(u) - \sin u]\ \nu(du).$$

For example, the function $h(u) := u\,\mathbf{1}_{\{u^2<c^2\}}$ that fully compensates only jumps of size $|u| < c$ would lead to

$$\begin{aligned} \delta^* &= \delta + \int [u\,\mathbf{1}_{\{u^2<c^2\}} - \sin u]\,\nu(du) \\ &= \delta + \frac{2\beta\gamma\Gamma(\alpha)\sin\frac{\pi\alpha}{2}\left[\alpha c^{1-\alpha} - \Gamma(2-\alpha)\sin\frac{\pi\alpha}{2}\right]}{\pi(1-\alpha)}, \qquad \alpha \neq 1 \end{aligned}$$

and (by L'Hôpital's rule) $\delta^* = \delta + \frac{2\beta\gamma}{\pi}[\log c + \gamma_e - 1]$, for $\alpha = 1$, where $\gamma_e \approx 0.577216$ denotes the Euler-Mascheroni constant.

Explicit Examples with $\alpha = 3/2$

Example $\mathsf{St}(3/2, 1, 1, 0)$**:**

For example, the fully-skewed Stable distribution with index $\alpha = 3/2$ and unit rate $\gamma = 1$ has Lévy measure

$$\nu(u) = \frac{3}{2\sqrt{2\pi}}\ u^{-5/2}, \qquad u > 0.$$

With the standard compensator $h(u) = \sin u$ it has offset $\delta = 0$, but with compensator $h(u) = \mathbf{1}_{\{u^2<1\}}$ it would have offset $\delta^* = 1 - 3/\sqrt{2\pi}$ and, with $h(u) = \mathbf{1}_{\{u^2<\epsilon^2\}}$, $\delta^*_\epsilon = 1 - 3/\sqrt{2\pi\epsilon}$. As $\epsilon \to 0$ we have $\delta^*_\epsilon \to -\infty$, a reminder that compensation is required for $\alpha \geq 1$.

Inverse Lévy Measure (ILM) for $\mathsf{St}(3/2, 1, 1, 0)$:

A random variable $X \sim \mathsf{St}(3/2, 1, 1, 0)$ may be generated by the Inverse Lévy Measure (ILM) algorithm of Wolpert and Ickstadt (1998a,b) by fixing $\epsilon > 0$, constructing a sequence τ_n of jumps of a standard Poisson process, setting

$$\begin{aligned}
\nu^+(u) &:= \nu\big([u,\infty)\big) \\
&= \int_u^\infty \frac{3}{2\sqrt{2\pi}}\, x^{-5/2}\, dx \\
&= u^{-3/2}/\sqrt{2\pi}, \\
\nu^-(t) &:= \sup[u > 0 : \nu^+(u) > t] \\
&= (2\pi)^{-1/3}\, t^{-2/3} = (t\sqrt{2\pi})^{-2/3} \\
\upsilon_n &:= \nu^-(\tau_n) = (\tau_n\sqrt{2\pi})^{-2/3} \\
X_\epsilon &:= \sum_{\upsilon_n > \epsilon} \upsilon_n - \int_\epsilon^\infty h(u)\, \nu(du) + \delta^* \\
&= \sum_{\upsilon_n > \epsilon} \upsilon_n + \delta^*_\epsilon \\
&= \sum_{\upsilon_n > \epsilon} \upsilon_n + \big[1 - 3/\sqrt{2\pi\epsilon}\,\big]. \qquad (11)
\end{aligned}$$

Alternatively, we may take $J_\epsilon \sim \mathsf{Po}\big(\epsilon^{-3/2}/\sqrt{2\pi}\big)$ and draw $\upsilon_j \overset{\text{iid}}{\sim} \mathsf{Pa}(3/2, \epsilon)$ from the Pareto distribution with density $(3/2)\epsilon^{3/2}\, u^{-5/2}\, \mathbf{1}_{(\epsilon,\infty)}(u)$, and set

$$X_\epsilon := \sum_{n=1}^{J_\epsilon} \upsilon_n + \big[1 - 3/\sqrt{2\pi\epsilon}\,\big]. \qquad (12)$$

Note the series in Eqns. (11, 12) would not converge without compensation; but *with* compensation, $X_\epsilon \to X_0 \sim \mathsf{St}(3/2, 1, 1, 0)$ as $\epsilon \to 0$.

A Stable *process* may be constructed similarly; if we draw $\sigma_n \sim \mathsf{Un}(0, 1)$ independently of τ_n, for example, we can set

$$X_\epsilon(t) := \sum_{\upsilon_n > \epsilon, \sigma_n \le t} \upsilon_n - t\left(\frac{3}{\sqrt{2\pi\epsilon}} - 1\right)$$

and again take $\epsilon \to 0$.

ILM for $\mathsf{St}(\alpha, \beta, \gamma, \delta)$:

A random variable $X \sim \mathsf{St}(\alpha, \beta, \gamma, \delta)$ may be generated by the ILM algorithm too, by fixing $\epsilon > 0$, constructing a sequence τ_n of jumps of a standard Poisson process and a sequence ρ_n of ± 1-valued independent random variables equal to $+1$ with probability $(1+\beta)/2$, setting

$$\begin{aligned}
\nu^+(u) &:= \nu\big((-\infty, -u] \cup [u, \infty)\big) \\
&= \frac{\gamma\, u^{-\alpha}}{\Gamma(1-\alpha)\cos\frac{\pi\alpha}{2}} \\
\nu^-(t) &:= \sup[u > 0 : \nu^+(u) > t] = \left(\frac{2\Gamma(\alpha)\gamma\sin\frac{\pi\alpha}{2}}{\pi t}\right)^{1/\alpha} \\
\upsilon_n &:= \nu^-(\tau_n) = \left(\frac{2\Gamma(\alpha)\gamma\sin\frac{\pi\alpha}{2}}{\pi \tau_n}\right)^{1/\alpha} \\
X_\epsilon &:= \sum_{\upsilon_n > \epsilon} \upsilon_n \rho_n - \gamma\beta\left(\int_\epsilon^\infty h(u)\, \nu(du) + \delta^*\right) + \delta \\
&= \sum_{\upsilon_n > \epsilon} \upsilon_n \rho_n - \frac{2\beta\gamma}{\pi(\alpha-1)}\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}\big(\epsilon^{1-\alpha} - 1\big) \\
&\qquad -\beta\gamma\left(\tan\frac{\pi\alpha}{2} + \frac{2}{\pi}\sin\frac{\pi\alpha}{2}\,\frac{\Gamma(1+\alpha)}{\alpha-1}\right) + \delta \\
&= \sum_{\upsilon_n > \epsilon} \upsilon_n \rho_n - \beta\gamma\left[\frac{2\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi(\alpha-1)}\epsilon^{1-\alpha} + \tan\frac{\pi\alpha}{2}\right] + \delta
\end{aligned}$$

A Stable *process* may be constructed similarly on $[0, T]$ by drawing $\sigma_n \sim \mathsf{Un}(0, T)$ independently of τ_n and setting

$$X_\epsilon(t) := T^{1/\alpha} \sum_{\upsilon_n > \epsilon, \sigma_n \le t} \upsilon_n \rho_n - \beta\gamma t\left[\frac{2\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi(\alpha-1)}\epsilon^{1-\alpha} + \tan\frac{\pi\alpha}{2}\right] + \delta t,$$

$0 \le t \le T$. A d-dimensional Stable *RF* may be constructed on $[0, T]^d$ by

$$\begin{aligned}
X_\epsilon[\phi] &:= T^{d/\alpha} \sum_{\upsilon_n > \epsilon} \upsilon_n \rho_n \phi(\sigma_n) \\
&\quad - \left\{\beta\gamma\left[\frac{2\alpha\Gamma(\alpha)\sin\frac{\pi\alpha}{2}}{\pi(\alpha-1)}\epsilon^{1-\alpha} + \tan\frac{\pi\alpha}{2}\right] - \delta\right\} \int_{[0,T]^d} \phi(s)\, d^d s,
\end{aligned}$$

for integrable $\phi \in L_1\left([0,T]^d\right)$. The distribution can be generalized somewhat by replacing γ and δ by nonnegative measures (on an arbitrary measure space), and α and β by measurable functions. This illustrates the advantage of parametrizing by a rate γ and location δ, rather than by scale.

Truncation Error Bounds:

For $\eta < \epsilon < 1$ the difference $X_\eta - X_\epsilon$ is given by

$$X_\eta - X_\epsilon = \sum_{\eta < |\upsilon_n| \le \epsilon} \upsilon_n \rho_n - \beta\gamma \left[\frac{2\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}}{\pi(\alpha-1)}\right]\left(\eta^{1-\alpha} - \epsilon^{1-\alpha}\right),$$

a mean-zero random variable with variance

$$\begin{aligned}
\mathsf{E}(X_\eta - X_\epsilon)^2 &= \int_{\eta<|u|<\epsilon} u^2 \nu(du) \\
&= \frac{2\gamma\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}}{\pi} \int_\eta^\epsilon u^{1-\alpha} \\
&= \frac{2\gamma\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}}{\pi(2-\alpha)} \left(\epsilon^{2-\alpha} - \eta^{2-\alpha}\right) \\
&\to \frac{2\gamma\Gamma(1+\alpha)\sin\frac{\pi\alpha}{2}}{\pi(2-\alpha)}\ \epsilon^{2-\alpha} \qquad \text{as } \eta \to 0,
\end{aligned}$$

so X_ϵ converges in L_2 as $\epsilon \to 0$ with the indicated L_2 truncation bound. Similar bounds apply to Stable processes and fields, with an additional factor of t or $\int \phi^2$ respectively.

2 Multivariate Stable Distributions

An $\mathbb{R}^d$-valued random vector X is Infinitely Divisible (ID) if, for any integer $n \in \mathbb{N}$, X may be written as the sum of n iid random variables. Familiar examples include Normal, Gamma, Negative Binomial, Poisson, α-stable, and many others. In 1936 Alexander Ya. Khinchine and Paul Lévy characterized such distributions as those whose log characteristic function may be written in what is now called Lévy-Khinchine form,

$$\begin{aligned}\log \chi(\omega) &\equiv \log \mathsf{E} \exp\left(i\omega' X\right) \\ &= \int_{\mathbb{R}^d} \left(e^{i\omega' u} - 1\right)\ \nu(du) \qquad (13a)\end{aligned}$$

for a measure $\nu(du)$ on $\mathbb{R}^d \backslash \{0\}$ satisfying the local L_1 condition

$$\int_{\mathbb{R}^d} (|u| \wedge 1)\ \nu(du) < \infty \qquad (13b)$$

or, more generally,

$$\log \chi(\omega) = \int_{\mathbb{R}^d} \left(e^{i\omega' u} - 1 - i\omega' h(u)\right)\ \nu(du) \qquad (14a)$$

for a measure $\nu(du)$ satisfying the less stringent local L_2 condition

$$\int_{\mathbb{R}^d} \left(|u|^2 \wedge 1\right)\ \nu(du) < \infty \qquad (14b)$$

and an arbitrary bounded measurable "compensator" function $h(u) = u + O(|u|^2)$ near $u \approx 0$. $\nu(du)$ is called the "Lévy measure."

The distribution is called *multivariate α-stable* if for some $0 < \alpha < 2$ the measure $\nu(du)$ is homogeneous of degree $-\alpha$, *i.e.*, satisfies $\nu(tA) = t^{-\alpha}\nu(A)$ for Borel sets $A \subset \mathbb{R}\backslash\{0\}$ and numbers $t > 0$. Upon changing to polar coordinates $u = r\sigma$ with $r \equiv |u| > 0$ and $\sigma \equiv u/|u| \in S^{d-1} = \left\{\sigma \in \mathbb{R}^d :\ |\sigma|^2 = 1\right\}$, such a measure can be written as

$$\nu(du) = c_\alpha \alpha r^{-\alpha-1}\, dr\, \Lambda(d\sigma) \qquad (15a)$$

for conventional normalizing constant

$$c_\alpha = \frac{2}{\pi}\, \Gamma(\alpha) \sin \frac{\pi\alpha}{2} \qquad (15b)$$

and finite positive Borel measure Λ on the unit d-sphere S^{d-1}.

2.1 The Uncompensated Case, $0 < \alpha < 1$

For $0 < \alpha < 1$ the measure $\nu(du)$ of (15a) satisfies the local L_1 condition (13b), so (13a) determines a valid log ch.f.

$$\begin{aligned}
\log \chi(\omega) &= \int_{\mathbb{R}^d} \left(e^{i\omega' u} - 1\right) \; \nu(du) \\
&= \int_{S^{d-1}} \left(c_\alpha \int_0^\infty \left[e^{i\omega'\sigma r} - 1\right] \alpha r^{-\alpha-1}\, dr \right) \; \Lambda(d\sigma) \\
&= -\int_{S^{d-1}} |\omega'\sigma|^\alpha \left[1 - i \tan\frac{\pi\alpha}{2} \operatorname{sgn}(\omega'\sigma)\right] \; \Lambda(d\sigma) \qquad \text{(16a)}
\end{aligned}$$

We denote the distribution of a random variable with this distribution, possibly offset by a constant $\delta \in \mathbb{R}^d$, by

$$X + \delta \sim \mathsf{St_A}(\alpha, \Lambda, \delta) \qquad \text{(16b)}$$

in a multivariate extension of Zolotarev's (A) parametrization. Note that Λ reflects *both* the intensity $\gamma \equiv \Lambda(S^{d-1}) \in \mathbb{R}^+$ and the noncentrality $\beta \equiv \int_{S^{d-1}} \sigma\Lambda(d\sigma)/\gamma \in B^d$, where $B^d = \{x \in \mathbb{R}^d : \; |x| \le 1\}$ is the closed unit ball in $\mathbb{R}^d$.

For example, in $d = 1$ dimension, the measure $\Lambda(d\sigma)$ is concentrated on just the two points $\{\pm 1\}$ of the 1-sphere S^0 and we may write it as

$$\Lambda(\{\sigma\}) = \gamma(1 + \sigma\beta)/2 = \begin{cases} \gamma(1+\beta)/2 & \text{for } \sigma = +1 \\ \gamma(1-\beta)/2 & \text{for } \sigma = -1 \end{cases}$$

for $\gamma \equiv \Lambda(S^0) > 0$ and $\beta \equiv \int \sigma\Lambda(d\sigma)/\gamma \in [-1, 1] = B^1$, giving log ch.f.

$$\begin{aligned}
\log \chi(\omega) &= \int_{\mathbb{R}} \left(e^{i\omega' u} - 1\right) \; \nu(du) \\
&= c_\alpha \int_{\mathbb{R}_+ \times S^0} \left(e^{i\omega'\sigma r} - 1\right) \; \alpha r^{-\alpha-1}\, dr\, \Lambda(d\sigma) \\
&= \gamma c_\alpha \int_0^\infty \left[\cos(\omega r) - 1 + i\beta \sin(\omega r)\right] \; \alpha r^{-\alpha-1}\, dr \\
&= -\gamma|\omega|^\alpha \left[1 - i\beta \tan\frac{\pi\alpha}{2} \operatorname{sgn}\omega\right],
\end{aligned}$$

so $X + \delta \sim \mathsf{St_A}(\alpha, \beta, \gamma, \delta)$ has the ordinary one-dimensional α-stable distribution.

If $X \sim \mathsf{St_A}(\alpha, \Lambda, \delta)$ has a d-variate α-stable distribution and $a \in \mathbb{R}^d$, then the linear combination $Y \equiv a'X$ has a univariate α-stable $\mathsf{St_A}(\alpha, \beta, \gamma, \delta_a)$ distribution with the same index α and with

$$\begin{aligned}\delta_a &= a'\delta \\ \gamma &= \int |a'\sigma|^\alpha \, \Lambda(d\sigma) \\ \beta &= \int |a'\sigma|^\alpha \, \operatorname{sgn}(a'\sigma) \, \Lambda(d\sigma)/\gamma.\end{aligned}$$

In particular, the univariate marginal distributions of the components are $X_j \sim \mathsf{St_A}(\alpha, \beta_j, \gamma_j, \delta_j)$ with $\gamma_j = \int |\sigma_j|^\alpha \Lambda(d\sigma)$ and $\beta_j = \int \sigma_j |\sigma_j|^{\alpha-1} \Lambda(d\sigma)/\gamma_j$. Similarly the matrix product AX for a $p \times d$ matrix A has a p-variate α-stable distribution with easily computed parameters.

2.1.1 Discrete Approximations, $0 < \alpha < 1$

Fix $0 < \alpha < 1$, two positive integers $d, p \in \mathbb{N}$, a $d \times p$ real full-tank matrix A, and p independent random variables $X_j \overset{\text{ind}}{\sim} \mathsf{St_A}(\alpha, \beta_j, \gamma_j, 0)$ for $\{\beta_j\} \subset [-1, 1]$ and $\{\gamma_j\} \subset \mathbb{R}_+$, and define $Y \in \mathbb{R}^d$ by

$$Y = AX, \qquad Y_i = \sum_{1 \le j \le p} A_{ij} X_j, \qquad 1 \le i \le d.$$

If we write the jth column of A as $\lambda_j \sigma_j$ for $\lambda_j > 0$ and $\sigma_j \in S^{d-1}$ for $1 \le j \le p$, then for $\omega \in \mathbb{R}^d$ we have

$$\begin{aligned}\mathsf{E}\left[e^{i\omega' Y}\right] &= \prod_j \mathsf{E}\left[e^{i(\omega'\sigma_j)\lambda_j X_j}\right] = \prod_j \left[e^{-\gamma_j |\omega'\sigma_j \, \lambda_j|^\alpha \left(1 - i\beta_j \tan\frac{\pi\alpha}{2} \operatorname{sgn}(\omega'\sigma_j \, \lambda_j)\right)}\right] \\ &= \exp\left\{-\sum_j |\omega'\sigma_j|^\alpha \left(1 - i\beta_j \tan\frac{\pi\alpha}{2} \operatorname{sgn}\omega'\sigma_j\right) \gamma_j \lambda_j^\alpha\right\} \\ &= \exp\left\{-\int_{S^{d-1}} |\omega'\sigma|^\alpha \left(1 - i \tan\frac{\pi\alpha}{2} \operatorname{sgn}\omega'\sigma\right) \Lambda(d\sigma)\right\}, \quad \text{w/} \\ \Lambda(d\sigma) &= \sum_{j \le p} \gamma_j \lambda_j^\alpha \left\{\frac{1+\beta_j}{2} \delta_{\{\sigma_j\}}(d\sigma) + \frac{1-\beta_j}{2} \delta_{\{-\sigma_j\}}(d\sigma)\right\}.\end{aligned}$$

Thus any $\mathsf{St_A}(\alpha, \Lambda, \delta)$ distribution can be approximated by a linear combination of p independent univariate α-Stables, constructed by approximating

Λ by a discrete measure concentrated at p pairs of antipodal points. If we require $\lambda_j \equiv 1$, by using unit vectors in S^{d-1} for the columns of A, then $\gamma \equiv \Lambda(S^{d-1}) = \sum \gamma_j$ and $\beta = \int_{S^{d-1}} \sigma \Lambda(d\sigma)/\gamma = \sum \beta_j \sigma_j \gamma_j / \gamma$.

2.2 The Compensated Case, $1 \leq \alpha < 2$

For $1 \leq \alpha < 2$ the measure $\nu(du)$ satisfies only the local L_2 condition (14b), so we must choose a bounded compensator function $h(u) = u + O(|u|^2)$ and apply (14a). A particularly convenient choice is $h(u) \equiv \sigma \sin r$ (where $r \equiv |u|$ and $\sigma \equiv u/|u|$), leading to log ch.f.

$$\begin{aligned}\log \chi(\omega) &= \int_{\mathbb{R}^d} \left(e^{i\omega' u} - 1 - i\omega'\sigma \sin r \right) \ \nu(du) &\text{(17a)}\\ &= c_\alpha \int_{\mathbb{R}_+ \times S^{d-1}} \left(e^{i\omega'\sigma r} - 1 - i\omega'\sigma \sin r \right) \ \alpha r^{-\alpha-1}\, dr\, \Lambda(d\sigma) \\ &= -\int_{S^{d-1}} \left\{ |\omega'\sigma|^\alpha \left[1 - i \tan\frac{\pi\alpha}{2} \operatorname{sgn}(\omega'\sigma) \right] - i \tan\frac{\pi\alpha}{2} \omega'\sigma \right\} \ \Lambda(d\sigma).\end{aligned}$$

(cf. (16a)). Extending Zolotarev's univariate (M) parametrization, we denote the distribution of a random variable with this distribution, possibly offset by a constant $\delta \in \mathbb{R}^d$, by

$$\begin{aligned} X + \delta \sim \mathsf{St}_\mathsf{M}(\alpha, \Lambda, \delta) &= \mathsf{St}_\mathsf{A}(\alpha, \Lambda, \delta^*) \qquad \text{for shifted offset} &\text{(17b)}\\ \delta^* &= \delta + \tan\frac{\pi\alpha}{2} \int_{S^{d-1}} \sigma \ \Lambda(d\sigma) = \delta + \gamma\beta \tan\frac{\pi\alpha}{2}. \end{aligned}$$

The (A) ch.f. (16a) has a singularity as $\alpha \to 1$ (because $\tan\frac{\pi\alpha}{2}$ does), but the (M) ch.f. (17a) is continuous there with limit

$$\log \chi(\omega) = -\int_{S^{d-1}} \left\{ |\omega'\sigma| - i\frac{2}{\pi}\, \omega'\sigma \ \log |\omega'\sigma| \right\} \ \Lambda(d\sigma) \qquad \text{(17c)}$$

for the multivariate Cauchy $\mathsf{St}_\mathsf{M}(1, \Lambda, 0)$ distribution.

2.3 Poisson Representation

A Poisson random measure $\mathcal{N}(dx)$ on a measurable space $(\mathcal{X}, \mathcal{F}, \mu)$ assigns disjoint random variables $\mathcal{N}(A_j) \sim \mathsf{Po}\big(\mu(A_j)\big)$ to disjoint sets $\{A_j\} \subset \mathcal{F}$ of finite measure $\mu(A_j)$, so the stochastic integrals

$$\mathcal{N}[\phi] = \int \phi(x)\mathcal{N}(dx) = \sum a_j \mathcal{N}(A_j)$$

of simple functions $\phi = \sum a_j \mathbf{1}_{\{A_j\}}$ (and hence, by a limiting argument, all ϕ with $(1 \wedge |\phi|) \in L_1(\mathcal{X}, \mathcal{F}, \mu)$) will have ch.f. $\chi(\omega) = \mathsf{E} \exp\big(i\omega\mathcal{N}[\phi]\big)$ given by

$$\prod\nolimits_j \big[\mathsf{E} e^{i\omega a_j \mathcal{N}(A_j)}\big] = \exp\Big(\sum (e^{i\omega a_j} - 1)\mu(A_j)\Big) = \exp\Big(\int (e^{i\omega\phi(x)} - 1)\mu(dx)\Big),$$

exactly the form of (16a) for $\phi(u) = u$, $\mathcal{X} = \mathbb{R}^d$, and $\mu = \nu$. It follows that any $\mathbb{R}^d$-valued random variable X with an ID distribution with Lévy measure $\nu(du)$ that satisfies the local L_1 condition (13b) may be represented in the form

$$X = \int_{\mathbb{R}^d} u\,\mathcal{N}(du) \tag{18}$$

for a Poisson random measure $\mathcal{N}(du) \sim \mathsf{Po}\big(\nu(du)\big)$. If $\nu(du)$ satisfies only the local L_2 condition (14b), compensation is required and we may write

$$X_h = \int_{\mathbb{R}^d} \big[u - h(u)\big]\ \mathcal{N}(du) + \int_{\mathbb{R}^d} h(u)\,\tilde{\mathcal{N}}(du)$$

where $\tilde{\mathcal{N}}(du) = \mathcal{N}(du) - \nu(du)$ is the "fully compensated" Poisson random measure associated with $\mathcal{N}(du)$ (Wolpert and Taqqu, 2005). Different choices of compensator $h(u)$ will lead to random variables X_{h_1}, X_{h_2} that differ only by a constant offset $\delta = [X_{h_2} - X_{h_1}] = \int [h_1(u) - h_2(u)]\nu(du)$. For the α-stable Lévy measure $\nu(du)$ of (15a), this Poisson representation leads to a link between the α-stables and multivariate extreme value theory.

2.4 MV EVT and MV α-Stables

Explicit expressions for the pdf and cdf of α-stable distributions are not available, except for two special cases (the Cauchy $\mathsf{St_A}(1, 0, \gamma, \delta)$ and the inverse Gaussian $\mathsf{St_A}(\frac{1}{2}, 1, \gamma, \delta)$) but the tails are always available explicitly: For $X \sim \mathsf{St_A}(\alpha, \beta, \gamma, \delta)$ and $u \gg (0 \vee \delta)$,

$$\begin{aligned}
\mathsf{P}[X > u] &\approx \mathsf{P}[\mathcal{N}\big((u - \delta, \infty)\big) > 0] \\
&= 1 - \exp\big(-\nu\big((u - \delta, \infty)\big)\big) \\
&= 1 - \exp\big(-c_\alpha[\gamma(1 + \beta)/2](u - \delta)^{-\alpha}\big) \\
&\approx c_\alpha[\gamma(1 + \beta)/2](u - \delta)^{-\alpha}
\end{aligned}$$

so

$$\lim_{u\to\infty} u^\alpha \; \mathsf{P}[X > u] = c_\alpha \gamma(1+\beta)/2. \tag{19}$$

and hence $g(X, \gamma(1+\beta)/2)$ has unit Fréchet tails for the function

$$g(x, y) = x\,|x|^{\alpha-1}/(c_\alpha y),$$

because, for $u \gg 0$, $Y \equiv g(X, \gamma(1+\beta)/2)$ has

$$\begin{aligned}
\mathsf{P}\{Y > u\} &= \mathsf{P}\left\{X\,|X|^{\alpha-1} > [c_\alpha\gamma(1+\beta)/2]\; u\right\} \\
&= \mathsf{P}\left\{X > [u\; c_\alpha\gamma(1+\beta)/2]^{1/\alpha}\right\} \approx 1/u.
\end{aligned}$$

Let $X \sim \mathsf{St_A}(\alpha, \Lambda, 0)$ have a d-variate α-Stable distribution; for $1 \le j \le d$ set

$$\begin{aligned}
\gamma_j &\equiv \int_{S^{d-1}} |\sigma_j|^\alpha\, \Lambda(d\sigma) \\
\beta_j &\equiv \int_{S^{d-1}} \sigma_j\, |\sigma_j|^{\alpha-1}\, \Lambda(d\sigma)/\gamma_j \\
\gamma_j^+ &\equiv \int_{S^{d-1}} \sigma_j^\alpha\, \mathbf{1}_{\{\sigma_j>0\}} \Lambda(d\sigma) = \gamma_j(1+\beta_j)/2 \\
Y_j &\equiv g(X_j, \gamma_j^+)
\end{aligned}$$

Then X has one-dimensional marginal distributions $X_j \sim \mathsf{St_A}(\alpha, \beta_j, \gamma_j, 0)$, and Y_j has unit Fréchet tails. Let "$X \ge 0$" denote the event $\cap_j (X_j \ge 0)$ that each component is nonnegative. For $u > 0$,

$$\begin{aligned}
\mathsf{P}\left\{Y \ge 0,\; \sum Y_j > u\right\} &= \mathsf{P}\left\{X \ge 0,\; \sum X_j^\alpha/\gamma_j^+ > c_\alpha u\right\} \\
&\approx \mathsf{P}\left\{\mathcal{N}\left(x \in \mathbb{R}_+^d :\; \sum x_j^\alpha/\gamma_j^+ > c_\alpha u\right) > 0\right\} \\
&\approx \nu\left\{x \in \mathbb{R}_+^d :\; \sum x_j^\alpha/\gamma_j^+ > c_\alpha u\right\} \\
&= u^{-1} \int_{S_+^{d-1}} \left\{\sum \sigma_j^\alpha/\gamma_j^+\right\}\; \Lambda(d\sigma) \\
&= d/u
\end{aligned}$$

where $S_+^{d-1} \equiv S^{d-1} \cap \mathbb{R}_+^d$ is the positive orthant of the unit sphere, and hence the extremal (or spectral) measure on S_+^{d-1} is

$$H(d\sigma) = \frac{1}{d} \left\{ \sum \sigma_j^\alpha / \gamma_j^+ \right\} \Lambda(d\sigma).$$

The projection of H onto the simplex Δ_d given for discrete spectra simply by mapping

$$\sigma \equiv \frac{w}{\sqrt{\sum w_j^2}} \in S_+^{d-1} \quad \rightleftarrows \quad w \equiv \frac{\sigma}{\sum \sigma_j} \in \Delta_d.$$

For absolutely continuous spectral measures the Jacobian for this $(d-1)$-dimensional transformation must be used. It might be helpful to compute that Jacobian, or to illustrate the whole process for $d = 2$, or both.

2.4.1 Example: $\mathbb{R}^2$

In $d = 2$ dimensions, $S_+^1 = \left\{ \sigma = (\sin\theta, \cos\theta) : \ 0 \le \theta \le \frac{\pi}{2} \right\}$ is conveniently parametrized by $\theta = \text{atan}(\sigma_1/\sigma_2)$. If Λ is absolutely continuous, then it has a density wrt the uniform distribution on S_+^{d-1},

$$\Lambda(d\sigma) = (2/\pi)\lambda(\theta) d\theta$$

and we have

$$\gamma_1 = \frac{2}{\pi} \int_0^{\pi/2} \sin(\theta)^\alpha \, \lambda(\theta) \, d\theta \qquad \gamma_2 = \frac{2}{\pi} \int_0^{\pi/2} \cos(\theta)^\alpha \, \lambda(\theta) \, d\theta$$

2.5 Estimation

Empirical estimates for Λ (on S^{d-1}) and H (on Δ_d) based on a sample $\{x_j\} \overset{\text{iid}}{\sim} \mathsf{St_A}(\alpha, \Lambda, 0)$ would be discrete probability measures with equal point masses at each of

$$\Lambda(d\sigma): \quad \sigma_j = \frac{x_j}{\|x\|_2} \qquad \left\{ x \in \mathbb{R}_+^d : \ \sum x_k^2 > u^2 \right\}$$

$$H(dw): \quad w_j = \frac{x_j \, |x_j|^{\alpha-1}/\gamma_j}{\sum x_k \, |x_k|^{\alpha-1}/\gamma_k} \qquad \left\{ x \in \mathbb{R}_+^d : \ \sum x_k^\alpha / \gamma_k > c_\alpha u^\alpha \right\}$$

The latter, of course, requires knowledge of α and $\{\gamma_k\}$, while the former does not. From $\hat{H}(dw)$ one could find estimates of $\kappa = \chi$, θ, or other functionals of $H(dw)$, or of the parameter(s) in a parametric family of submodels.

A Appendix: Popular Parametrizations

Lévy (1925, Eqn. (103) on $p.\,255$) introduced the α-Stable family of distributions as those with log ch.f. of the form (in his notation)

$$\psi(t) = -(c_0 + c_1 j)|t|^\alpha, \tag{20}$$

where $j = i \operatorname{sgn} t$. Since then the α-Stable family has been parametrized in a bewildering variety of ways by a wide variety of authors (for an amusing account, see Hall, 1981). Nolan (1999, $p.\,3$) attributes to Samorodnitsky and Taqqu (1994, $p.\,5$) the location-scale family "$X \sim \mathsf{St}(\alpha, \beta, \gamma, \delta)$" built on parametrization (A) of Zolotarev (1986, $p.\,9$) with

$$\begin{aligned} \mathsf{E}e^{i\omega X} &= \exp\left\{-\gamma^\alpha|\omega|^\alpha\left[1 - i\beta\tan\frac{\pi\alpha}{2}\operatorname{sgn}\omega\right] + i\delta\omega\right\} \\ \mathsf{E}X &= \delta, \end{aligned} \tag{21}$$

and calls it the "S1" parametrization. Because S1 is discontinuous at $\alpha = 1$, he himself prefers what he calls the "S0" parametrization (Nolan, 1999, $p.\,4$), recommended earlier by Cheng and Liu (1997, Eqn. (11)),

$$\begin{aligned} \mathsf{E}e^{i\omega X} &= \exp\left\{-\gamma^\alpha|\omega|^\alpha + i\beta\omega\tan\frac{\pi\alpha}{2}\left(\gamma^\alpha|\omega|^{\alpha-1} - \gamma\right) + i\delta\omega\right\} \\ \mathsf{E}X &= \delta - \beta\gamma\tan\frac{\pi\alpha}{2} \end{aligned} \tag{22}$$

which he describes as a "variation of Zolotarev's (M) parametrization." Zolotarev (1986, $p.\,11$) introduced his famous continuous (M) parametrization

$$\begin{aligned} \mathsf{E}e^{i\omega X} &= \exp\left\{\gamma\left[i\delta - |\omega|^\alpha + i\omega\left(|\omega|^{\alpha-1} - 1\right)\beta\tan\frac{\pi\alpha}{2}\right]\right\} \\ \mathsf{E}X &= \gamma\delta - \beta\gamma\tan\frac{\pi\alpha}{2} \end{aligned} \tag{23}$$

to remove the "disagreeable feature" of "discontinuities at all points of the form $\alpha = 1,\ \beta \neq 0$" in Eqn. (21) (note $\tan\frac{\pi\alpha}{2} \to +\infty$ as $\alpha \uparrow 1$, $\tan\frac{\pi\alpha}{2} \to -\infty$ as $\alpha \downarrow 1$); S0 (Eqn. (22)) is the location-scale family built on the standard ($\gamma = 1,\ \delta = 0$) case of (M) (Eqn. (23)). Although in (Wolpert, 2002, $p.\,2$) I used parametrization Eqn. (22), I now prefer Zolotarev's (M) or yet another slight variation,

$$\begin{aligned} \mathsf{E}e^{i\omega X} &= \exp\left\{i\delta\omega - \gamma|\omega|^\alpha + i\beta\gamma\tan\frac{\pi\alpha}{2}\omega\left(|\omega|^{\alpha-1} - 1\right)\right\} \\ \mathsf{E}X &= \delta - \beta\gamma\tan\frac{\pi\alpha}{2}. \end{aligned} \tag{24}$$

This is identical to Zolotarev's (M), except the location parameter here called δ would be his $\gamma\delta$; the form of Eqn. (24) leads to slightly more general α-Stable random fields (see Section A.1.1 below) assigning independent Stables to disjoint sets. In each of Eqns. (21–24) the ch.f. is valid for $\alpha \neq 1$ (take limits for $\alpha \to 1$) and the mean is only valid for $1 < \alpha < 2$ (it's undefined or infinite for $\alpha \leq 1$). Note too that $\tan\frac{\pi\alpha}{2} < 0$ for $\alpha \in (1,2)$, so each mean is positive for $\beta > 0$, $\delta = 0$.

In the fully-skewed case ($\beta = 1$), Zolotarev's (A) parametrization with zero offset $\delta = 0$ is sometimes written in complex form

$$\mathsf{E}e^{i\omega X} = e^{-\gamma|\omega|^\alpha(1-i\beta\tan\frac{\pi\alpha}{2}\operatorname{sgn}\omega)} = e^{-\gamma^*(-i\omega)^\alpha}$$

where $\gamma^* = \gamma\sec\frac{\pi\alpha}{2}$.

The `R` package `fBasics` offers functions `[dpqr]stable(...,pm)` returning (respectively) the density, distribution, quantile, and random draws from the α-Stable distribution in a parametrization indexed by an integer `pm` taking one of three values (the package is available from `CRAN`). The value `pm=0` indicates Nolan's "S0" continuous parametrization (Eqn. (22)), `pm=1` for Samorodnitsky and Taqqu's discontinuous "S1" parametrization (Eqn. (21)), or `pm=2` for Nolan rather silly "S2" parametrization intended to have the scale parameter reduce to the traditional ones for the two cases of Cauchy ($\alpha = 1$) and Gaussian ($\alpha = 2$). To achieve $\mathsf{St}(\alpha,\beta,\gamma,\delta)$ in (my preferred) parametrization Eqn. (24), use:

`pm=0`	`alpha=`α	`beta=`β	`gamma=`$\gamma^{1/\alpha}$	`delta=`$\delta - \beta\tan\frac{\pi\alpha}{2}(\gamma - \gamma^{1/\alpha})$
`pm=1`	`alpha=`α	`beta=`β	`gamma=`$\gamma^{1/\alpha}$	`delta=`$\delta - \beta\gamma\tan\frac{\pi\alpha}{2}$

Since Eqn. (24) and the continuous parametrizations of Cheng and Liu (1997), Zolotarev (1986) and Nolan (1998b) all agree for the standard case of $\gamma = 1$ and $\delta = 0$, it is probably safest to standardize when switching among them. If $Z \sim \mathsf{St}(\alpha,\beta,1,0)$ (in any of these parametrizations) with ch.f.

$$\mathsf{E}e^{i\omega Z} = e^{-|\omega|^\alpha + i\beta\tan\frac{\pi\alpha}{2}\omega(|\omega|^{\alpha-1}-1)}$$

then the affine transformation $X = mZ + b$ has the $\mathsf{St}(\alpha,\beta,\gamma,\delta)$ distribution in each of these parametrizations with α unchanged, β replaced by $\beta\operatorname{sgn} m$, and the other two parameters given by:

Zolotarev (M):	$\gamma = \lvert m\rvert^\alpha$	$\delta = b$
Nolan S0:	$\gamma = \lvert m\rvert$	$\delta = b$
Eqn. (24):	$\gamma = \lvert m\rvert^\alpha$	$\delta = b + \beta\tan\frac{\pi\alpha}{2}\, m\,(\lvert m\rvert^{\alpha-1} - 1)$
Sam. & Taqqu S1:	$\gamma = \lvert m\rvert$	$\delta = b - \beta\tan\frac{\pi\alpha}{2}\, m$

A.1 The Ideas Behind the Popular Parametrizations

The simplest formula, Eqn. (21), is a variation on the original equations of Lévy. Its advantage of simplicity is offset by the disadvantage that things go crazy when the index α gets close to one (because of the infinite discontinuity of $\tan\frac{\pi\alpha}{2}$ in Eqn. (21)); Cheng and Liu's parametrization Eqn. (22) was motivated by a painfully explicit search for continuity near $\alpha \approx 1$ (they seem unaware that Zolotarev had already found a continuous parametrization). Zolotarev's (M) choice Eqn. (23) has exactly the simple Lévy-Khinchine form

$$\mathsf{E}e^{i\omega X} = \exp\left\{ i\gamma\delta\omega + \int_{\mathbb{R}} \left[e^{i\omega u} - 1 - i\omega \sin u \right] \nu(du) \right\} \tag{25a}$$

with compensator $h(u) = \sin u$ and Lévy measure

$$\nu(du) = \frac{\alpha\gamma}{\pi}\Gamma(\alpha)\sin\frac{\pi\alpha}{2}|u|^{-\alpha-1}\big(1 + \beta\,\mathrm{sgn}\,u\big)\,du. \tag{25b}$$

It has the odd feature that the offset term $\gamma\delta$ lends γ the dual role of determining the rate for both the drift and the random process. Eqn. (24) differs only in this one respect, replacing the drift $\gamma\delta$ in Eqn. (25a) with δ. Nolan's favorite S0 (Eqn. (22)) begins with the standard ($\gamma = 1$, $\delta = 0$) α-Stable in Zolotarev's (M) parametrization, then builds a location-scale family with the remaining two parameters; this makes scaling simple at the expense of more complex formulas for sums of independent α-Stable variables and especially for stochastic processes and random fields, as we now see.

A.1.1 Lévy Flights and Random Measures

My preference Eqn. (24) differs from Zolotarev's Eqn. (23) only in the drift (δ instead of $\gamma\delta$). It has the very nice property that you can construct a *random measure* $X(dx)$ over an arbitrary measure space $(\mathcal{X}, \mathcal{F})$ with *measure*-valued parameters $\gamma(dx)$ and $\delta(dx)$ and a function-valued parameter $\beta(x)$, so that for disjoint sets $A_i \subset \mathcal{X}$ the random variables $X(A_i)$ are independent with stable distributions $X(A_i) \sim \mathsf{St}(\alpha, \beta_i, \gamma_i, \delta_i)$ where $\beta_i = \int_{A_i} \beta(x)\,\gamma(dx)/\gamma(A_i)$, $\gamma_i = \gamma(A_i)$, and $\delta_i = \delta(A_i)$. More generally, for $\phi \in L_1\big(\mathcal{X},\ \delta(dx)\big) \cap L_1\big(\mathcal{X},\ \gamma(dx)\big) \cap L_\alpha\big(\mathcal{X},\ \gamma(dx)\big)$ the stochastic integral $X[\phi] := \int_{\mathcal{X}} \phi(x)\,X(dx)$ is well-defined with an α-Stable distribution $X[\phi] \sim \mathsf{St}(\alpha_\phi, \beta_\phi, \gamma_\phi, \delta_\phi)$ with parameters

$$\begin{aligned}
\alpha_\phi &= \alpha \\
\beta_\phi &= \int_{\mathcal{X}} \beta(x) |\phi(x)|^\alpha \operatorname{sgn} \phi(x)\, \gamma(dx) \Big/ \int_{\mathcal{X}} |\phi(x)|^\alpha\, \gamma(dx) \\
\gamma_\phi &= \int_{\mathcal{X}} |\phi(x)|^\alpha\, \gamma(dx) \\
\delta_\phi &= \int_{\mathcal{X}} \phi(x)\, \delta(dx) + \tan \frac{\pi\alpha}{2} \int_{\mathcal{X}} \beta(x)\phi(x) \big[|\phi(x)|^{\alpha-1} - 1 \big] \gamma(dx).
\end{aligned}$$

For example, with $\mathcal{X} = \mathbb{R}$, constant $\beta(x) \equiv \beta$, and each of the measures $\gamma(dx) = \gamma\, dx$ and $\delta(dx) = \delta\, dx$ proportional to Lebesgue measure, the stochastic process

$$X_t := \begin{cases} X\{(0,t]\} & t \ge 0 \\ -X\{(t,0]\} & t < 0 \end{cases}$$

has stationary independent increments $X_t - X_s \sim \mathsf{St}\big(\alpha, \beta, \gamma|t-s|, \delta(t-s)\big)$. For $\alpha = 2$ it is Brownian motion with drift, while for $\alpha \in (0, 2)$ it is often called the α-Stable Lévy Flight. Similar multivariate processes and random fields can be constructed with $\mathsf{St}_\mathsf{M}(\alpha, \Lambda, \delta)$ marginal distributions.

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