Conditional Probability, Intro Markov Chains

I. $P_{ij}^{n,n+1} = \mathsf{P}[X_{n+1} = s_j | X_n = s_i] \stackrel{?}{=} P_{ij}$
   A. Properties: $P_{ij} \geq 0$; $\sum_j P_{ij} \equiv 1$
   B. State space: Discrete set $\mathcal{S} = \{s_i\}$, for now...

II. $\pi_j^n = \sum_{i=0}^N \pi_i^0 \, [P^n]_{ij} = [\pi^0 \, P^n]_j$
   A. $\pi_j^n \to \pi_j$, where $\pi P = \pi$
   B. Metropolis: Set $\alpha_{ij} = \min\left\{\frac{\pi_j Q_{ji}}{\pi_i Q_{ij}}, 1\right\}$, $P_{ij} = Q_{ij} \, \alpha_{ij}$, $i \neq j$; then $\pi = \pi$. For special case of $Q_{ij} = 1/2$, $j = i \pm 1$, $\alpha_{ij} = \min(\pi_i, \pi_j)/\pi_i$.

III. Accessible, Communicating
   A. $j$ "Accessible from $i$" ($i \to j$) if $\exists n \ni P_{ij}^n \neq 0$ (draw directed graph)
   B. $i, j$ "Communicate" ($i \leftrightarrow j$) if $\exists n, m \ni P_{ij}^n > 0, P_{ji}^m > 0$
   C. $A$ is "Irreducible" if $\forall \, i, j$, $i \leftrightarrow j$
   D. State $i$ is "Periodic with period $d(i)$" if $d(i)$ is the GCD of $\{n : [P^n]_{ii} > 0\}$; if $d(i) = 1$, then $i$ is "aperiodic."
      1. $A$ is "Aperiodic" if $d(i) = 1$ for every $i$
      2. If $i \leftrightarrow j$, then $d(i) = d(j)$
   E. Theorem: State $i$ is "Recurrent" if $\sum_n P_{ii}^n = \infty$
      1. $P_{ij}^n = \sum_{k=0}^n f_{ij}^k \, P_{jj}^{n-k}$, $n \geq 0$ & $i \neq j$ *or* $n \geq 1$
      2. $P_{ij}(s) = \sum_{n=0}^\infty P_{ij}^n \, s^n$, Generating Function
      3. $F_{ij}(s) = \mathsf{E}_i[s^{\tau_j}] = \sum_{n=1}^\infty f_{ij}^n \, s^n$
      4. $P_{ij}(s) = F_{ij}(s) P_{jj}(s) + \delta_{ij}$, so $P_{ii}(s) = \frac{1}{1 - F_{ii}(s)}$; let $s \to 1$.
      5. $\sum_n s^n P^n = [I - sP]^{-1}$, so just solve $[I - sP]v = e^i$

IV. Some linear algebra for finite-state Markov chains:
   A. To find a stationary distribution $\pi P = \pi \Rightarrow [I - P']\pi' = 0$, let $B$ be an $n \times n$ matrix with rows $2 \to n$ equal to those of $I - P$, but row 1 $[1, 1, \cdots, 1]$, and solve $B'\pi' = e^1$ to get normalized stationary distribution.
   B. To find the probability of hitting $j$ from $i$ in time $t$ let $B$ be an $n \times n$ matrix equal to $P$ except for row $j$, which is set to $e^j$; then the answer is $\mathsf{P}_i[\tau_j \leq n] = [B^n]_{ij}$.
   C. To find the expected time $\tau_j$ until hitting $j$ from $i$, let $B$ be an $n \times n$ matrix equal to $[I - P]$ with the $j^{th}$ row and column zeroed out, except $B_{jj} = 1$; let $b_i = 1$ for $i \neq j$, $b_j = 0$, and solve the $n \times n$ system $B \, x = b$ for $x$; $\mathsf{E}_i[\tau_j] = x_i$ is the expected wait (starting at $i$) until hitting $j$.

V. Some Martingales
   A. For any $u : S \to \mathbb{R}$ write $u_i = u(s_i)$, $[I - P]u(i) = \sum_j P_{ij}(u_i - u_j)$; then $Y_t = u(X_t) + \sum_{s<t} [I - P]u(X_s)$ is a martingale for every $u$.
   B. Call $u$ *harmonic* if $[I - P]u = 0$, *subharmonic* if $[I - P]u \leq 0$, and *superharmonic* if $[I - P]u \geq 0$; then $u(X_t)$ is a martingale, submartingale, and supermartingale, respectively (here "$u \geq 0$" means that each component $u_i$ is nonnegative).
   C. Random walk: If $P_{ij} = p$ for $j = i{+}1$, $P_{ij} = q$ for $j = i{-}1$, and $P_{ii} = r = 1 - p - q$, and if $u_i = \alpha + \beta i + \gamma i^2$, then $[I - P]u_i = (\beta - 2\gamma i)(p - q) - \gamma(p + q)$, while for $u_i = s^i$, $[I - P]u_i = s^{i-1}(1 - s)(ps - q)$; thus $u_i = \alpha + \beta X_t$ and $(X_t)^2 - 2pt$ are harmonic for symmetric random walks and $\alpha + \beta(q/p)^{X_t}$ is harmonic for asymetric ones.
   D. Success Runs: If $P_{ij} = p_i$ for $j = i{+}1$, $P_{i0} = q_i = 1 - p_i$, with each $p_i > 0$, then set $u_0 = 1$ and $u_{i+1} = (u_i - q_i)/p_i$ to get a harmonic $u$.
   E. Question: What functions $u(x)$ are harmonic for 2-dimensional and $p-$dimensional symmetric random walks in the plane and $\mathbb{R}^p$???