Continuous-Time Markov Chains
Karlin & Taylor, *A First Course in Stochastic Processes, pp 145–152*

Return to Zero for Birth/Death Processes

The probability $F_i(t)$ of return to state 0 at or before time t, starting from $X_0 = i$, can also be studied by differential equation methods:

$$\begin{aligned} F_i(t+\epsilon) &= \epsilon\lambda_i F_{i+1}(t) + \epsilon\mu_i F_{i-1}(t) + [1-\epsilon(\lambda_i+\mu_i)]F_i(t) + o(\epsilon) \\ F_i'(t) &= \lambda_i F_{i+1}(t) + \mu_i F_{i-1}(t) - (\lambda_i+\mu_i)F_i(t) \end{aligned}$$

In the limit as $t \to \infty$, the probability F_i of *ever* returning to zero satisfies

$$\begin{aligned} 0 &= -\lambda_i[F_i - F_{i+1}] + \mu_i[F_{i-1} - F_i] \\ [F_i - F_{i+1}] &= [F_{i-1} - F_i](\frac{\mu_i}{\lambda_i}) = [1-F_1](\frac{\mu_1\mu_2\cdots\mu_i}{\lambda_1\lambda_2\cdots\lambda_i}) \\ 1 - F_j &= \sum_{i=0}^{j-1}[F_i - F_{i+1}] = [1-F_1]\sum_{i=0}^{j-1}(\frac{\mu_1\mu_2\cdots\mu_i}{\lambda_1\lambda_2\cdots\lambda_i}). \end{aligned}$$

If $\sum_{i=0}^{\infty}\rho_i = \infty$, where $\rho_i \equiv \big(\frac{\mu_1\mu_2\cdots\mu_i}{\lambda_1\lambda_2\cdots\lambda_i}\big)$, then $[1-F_1] = 0$ and $F_j \equiv 1$; if the ρ_i are summable, then

$$F_j = \frac{\sum_{i=j}^{\infty}\rho_i}{\sum_{i=0}^{\infty}\rho_i}$$

For the linear birth/death process $\rho_i = (\frac{\delta}{\beta})^i$, $\sum_{i=j}^{\infty}\rho_i = \frac{(\frac{\delta}{\beta})^j}{1-\frac{\delta}{\beta}}$, and so

$$F_j = \begin{cases} (\frac{\delta}{\beta})^j & \text{if } \delta < \beta \\ 1 & \text{if } \delta \ge \beta. \end{cases}$$

In case $\sum_{i=0}^{\infty}\rho_i = \infty$ the process is certain to return to zero, but when? The expectation of the first hitting-time of zero, $G_i = \mathsf{E}[T_0 \mid X_0 = i]$, satisfies $G_0 = 0$ and, for $i > 0$,

$$\begin{aligned} G_i &= \frac{1}{\lambda_i+\mu_i} + (\frac{\lambda_i}{\lambda_i+\mu_i})G_{i+1} + (\frac{\mu_i}{\lambda_i+\mu_i})G_{i-1}, \\ \lambda_i[G_{i+1} - G_i] &= \mu_i[G_i - G_{i-1}] - 1 \\ [G_{i+1} - G_i] &= \frac{\mu_i}{\lambda_i}[G_i - G_{i-1}] - \frac{1}{\lambda_i} \\ &= \frac{\mu_i\mu_{i-1}\cdots\mu_1}{\lambda_i\lambda_{i-1}\cdots\lambda_1}[G_1 - G_0] - [\frac{1}{\lambda_i} + \frac{\mu_i}{\lambda_i\lambda_{i-1}} + \cdots + \frac{\mu_i\cdots\mu_2}{\lambda_i\cdots\lambda_1}] \\ &= \rho_i G_1 - \sum_{k=1}^{i}\frac{\rho_i}{\lambda_k\rho_k} \qquad \big(\text{recall } \rho_i \equiv (\tfrac{\mu_1\mu_2\cdots\mu_i}{\lambda_1\lambda_2\cdots\lambda_i})\big), \text{ so} \\ \frac{G_{i+1} - G_i}{\rho_i} &= G_1 - \sum_{k=1}^{i}\frac{1}{\lambda_k\rho_k} \to 0. \quad \text{Thus} \\ G_1 &= \sum_{k=1}^{\infty}\frac{1}{\lambda_k\rho_k}, \qquad \text{and, more generally,} \\ G_j &= \sum_{i=1}^{j}\sum_{k=i}^{\infty}\frac{\rho_{i-1}}{\lambda_k\rho_k}. \end{aligned}$$

For the linear birth/death process with immigration, $\frac{\rho_{i-1}}{\lambda_k\rho_k} = \frac{\lambda_i\lambda_{i+1}\cdots\lambda_{k-1}}{\mu_i\mu_{i+1}\cdots\mu_k} = \frac{\Gamma(\frac{a}{\beta}+k)\,(i-1)!}{a\;\Gamma(\frac{a}{\beta})\;k!}\Big(\frac{\beta}{\delta}\Big)^{k-i+1}$, so

$$G_1 = \sum_{k=1}^{\infty}\frac{\Gamma(\frac{a}{\beta}+k)}{a\;\Gamma(\frac{a}{\beta})\;k!}\Big(\frac{\beta}{\delta}\Big)^k$$

$$= \frac{1}{a}\Big(\big(1 - \frac{\beta}{\delta}\big)^{-a/\beta} - 1\Big) \qquad \text{(the Neg. Binom. sum)}$$
$$\to \frac{-1}{\beta}\log\big(1 - \frac{\beta}{\delta}\big) \text{ as } a \to 0. \qquad \text{(the text's answer for } a = 0.\text{)}$$

Finite State Markov Chains

The study of continuous-time Markov chains with finite state space $\mathcal{S} = \{0, ..., N\}$ is in some ways easier than the general case, because the tools of linear algebra can be brought to bear. It is also important in applications.

Let $P_{ij}(t)$ be the transition function for a finite-state Markov chain and $Q_{ij} = P'_{ij}(0)$ the infinitessimal transition matrix, so

$$P_{ij}(\epsilon) = \begin{cases} \epsilon Q_{ij} + o(\epsilon) & \text{if } i \neq j \\ 1 - \epsilon \sum_{k \neq i} Q_{ik} + o(\epsilon) & \text{if } i = j. \end{cases}$$

For the finite-state case, the Kolmogorov Backward and Forward equations are:

$$P'_{ij}(t) = \sum_{k=0}^{N} Q_{ik} P_{kj}(t) \qquad (BE)$$
$$= \sum_{k=0}^{N} P_{ik}(t) Q_{kj}; \qquad (FE)$$

recall that

$$Q_{ij} = P'_{ij}(0)$$

is just the rate of jumping from state i to j, if $i \neq j$. Since $1 = \sum_j P_{ij}(t)$ for all t, $0 = \sum_j P'_{ij}(t)$ for any t and in particular for $t = 0$, so $-Q_{ii} = \sum_{k \neq i} Q_{ik}$ is just the total rate of jumping away from state i.

Any $(N+1) \times (N+1)$ matrix Q has $N+1$ real or complex eigenvalues λ_i, the $N+1$ solutions of the polynomial equation $p(\lambda) = \det[Q - \lambda I] = 0$. If the λ_i are all distinct, and often, even if they are not, Q can be put in the Jordan canonical form:

$$Q = U \Lambda U^{-1} \qquad (JCF)$$

where U is an $(N+1) \times (N+1)$ matrix whose columns u^i are (right) eigenvectors for Q, $V = U^{-1}$ is the matrix inverse of U whose rows v^j are left eigenvectors for Q, and Λ is a diagonal matrix of eigenvalues $\lambda_i = \Lambda_{ii}$ of Q. Some matrices cannot *quite* be put into form (JCF), if the $\{\lambda_i\}$ are not distinct; even in that case the Jordan canonical form can still be used, but we must allow the possibility that $\Lambda_{i,j} = 1$ for some i, j such that $\lambda_i = \lambda_j$ and $|i - j| = 1$. We will ignore that possibility in the sequel.

The Kolmogorov Forward Equation (FE) with initial condition $P_{ij}(0) = \delta_{ij}$ can be solved for $P_{ij}(t)$. Formally the solution to the matrix equation $P'(t) = P(t)Q$ is the familiar $P(t) = P(0)\exp(tQ)$; when $\mathcal{S}$ is finite we can make sense of that prescription in at least two different ways:

$$P(t) = \sum_{n=0}^{\infty} \frac{t^n Q^n}{n!} \qquad \text{(power-series)}$$
$$= e^{tQ}$$
$$= \big[U\, e^{t\Lambda}\, V\big]_{ij} \qquad \text{(recall } V = U^{-1}\text{)}$$
$$= \sum_{k=0}^{N} e^{t\lambda_k}\, u_k\, v'_k \qquad \text{(eigen-series)}$$

either in an infinite series or in an eigen-vector expansion. Note that it is the matrix product Q^n appearing in the infinite series, defined recursively by $Q^n_{ij} = \sum_{k=0}^{N} Q^{n-1}_{ik} Q_{kj}$. When $\mathcal{S}$ is infinite the power series expansion might not converge and the eigenvalue analysis of transition functions is more subtle.