BROWNIAN BRIDGE

S Karlin & J Taylor, *A Second Course in Stochastic Processes, ch 15*
G Kallianpur, *Stochastic Filtering Theory, ch 5*

Let $w(t)$ be a standard Brownian Motion, *i.e.*, a continuous-path Gaussian stochastic process with mean $\mathsf{E}[w(t)] \equiv 0$ and covariance $\mathsf{E}[w(s)w(t)] = (s \wedge t)$, the minimum of s and t (for nonnegative s, t). Now define two processes for $0 \le t \le 1$ by:

$$X(t) \equiv w(t) - t w(1) \tag{1}$$

$$Y(t) \equiv (1-t) w\left(\frac{t}{1-t}\right) \tag{2}$$

It is easy to see that both $X(\cdot)$ and $Y(\cdot)$ are continuous-path mean-zero Gaussian stochastic processes (because $w(\cdot)$ is); their covariances for $0 \le s \le t \le 1$ are:

$$\begin{aligned}
\mathsf{E}[X(s)X(t)] &= \mathsf{E}\big(w(s) - s w(1)\big)\big(w(t) - t w(1)\big) \\
&= s - st - st + st \\
&= (s \wedge t) - st && (3a) \\
\mathsf{E}[Y(s)Y(t)] &= (1-s)(1-t)\left(\frac{s}{1-s} \wedge \frac{t}{1-t}\right) \\
&= (1-t)s \\
&= (s \wedge t) - st && (3b)
\end{aligned}$$

Each of these is the **Brownian Bridge**, a zero-mean continuous-path Gaussian process with covariance $(s \wedge t) - st$.

Properties

For $0 \le t \le 1$ let $\mathcal{F}_t = \sigma\{Y_s : s \le t\}$ be the σ-algebra generated by the Brownian Bridge; from (2) it is clear that, for $t < 1$ and $0 < \epsilon < 1 - t$, the conditional distribution of $Y(t+\epsilon)$, given the past $\mathcal{F}_t$, is Gaussian with nonzero *conditional* predictive mean

$$\begin{aligned}
\mathsf{E}[Y(t+\epsilon) | \mathcal{F}_t] &= \mathsf{E}\Big[(1-t-\epsilon)\big[w\left(\frac{t}{1-t}\right) + w\left(\frac{t+\epsilon}{1-t-\epsilon}\right) - w\left(\frac{t}{1-t}\right)\big]\Big|\mathcal{F}_t\Big] \\
&= (1-t-\epsilon)\big[w\left(\frac{t}{1-t}\right)\big] \\
&= \frac{1-t-\epsilon}{1-t}\big[Y(t)\big] = Y_t - \epsilon\,\frac{Y_t}{1-t} && (4)
\end{aligned}$$

and conditional predictive variance

$$\begin{aligned}
\mathsf{E}[(Y(t+\epsilon) - \mathsf{E}[Y(t+\epsilon)|\mathcal{F}_t])^2 | \mathcal{F}_t] &= \mathsf{E}\Big[(1-t-\epsilon)^2\big(w\left(\frac{t+\epsilon}{1-t-\epsilon}\right) - w\left(\frac{t}{1-t}\right)\big)^2\Big|\mathcal{F}_t\Big] \\
&= (1-t-\epsilon)^2\left(\frac{t+\epsilon}{1-t-\epsilon} - \frac{t}{1-t}\right) \\
&= (1-t-\epsilon)^2\left(\frac{(t+\epsilon)(1-t) - t(1-t-\epsilon)}{(1-t)(1-t-\epsilon)}\right) \\
&= \frac{\epsilon(1-t-\epsilon)}{(1-t)} = \epsilon - \frac{\epsilon^2}{1-t} && (5)
\end{aligned}$$

By equation (4) we see that $Y(t)$ is not a martingale, but that

$$W_t \equiv Y(t) + \int_0^t \frac{1}{1-s} Y(s)\, ds$$

is a martingale. In fact W_t is a continuous-path Gaussian martingale, starting at $W_0 = 0$, with mean zero. From the martingale property we must have $\mathsf{E}[W_s W_t] = \mathsf{E}[W_{s\wedge t}^2]$ (since, for $s < t$, $\mathsf{E}[W_s(W_t - W_s)|\mathcal{F}_s] = 0$), while the variance is given by

$$\begin{aligned}
\mathsf{E}[W_t^2] &= t(1-t) + 2\int_0^t \frac{u-ut}{1-u}\, du + 2 \iint_{0<u<v<t} \frac{u-uv}{(1-u)(1-v)}\, du\, dv \\
&= t - t^2 + 2\int_0^t \frac{u(1-t) + u(t-u)}{1-u}\, du \\
&= t, \text{ so} \\
\mathsf{E}[W_s W_t] &= s \wedge t.
\end{aligned}$$

Thus we see that $W(t)$ is just a Wiener process and we have the (third!) representation of $Y(t)$ as the solution to a *Stochastic Integral Equation*:

$$Y(t) = 0 - \int_0^t \frac{1}{1-s} Y(s)\, ds + W_t, \tag{6}$$

from which it is again apparent that the conditional mean and variance of $Y(t+\epsilon)$, given $\mathcal{F}_t$, are (respectively)

$$\begin{aligned}
\mathsf{E}[Y(t+\epsilon)|\mathcal{F}_t] &= Y(t) - \epsilon \frac{Y(t)}{1-t} + o(\epsilon) \\
\mathsf{V}[Y(t+\epsilon)|\mathcal{F}_t] &= \epsilon + o(\epsilon),
\end{aligned}$$

just as in equations (4) and (5). This is our first example of a **Diffusion**; it is common to write this in differential form as

$$dY(t) = -\frac{1}{1-t} Y(t)\, dt + dW_t,$$

and to write more general diffusions in the similar form

$$dX_t = \mu(t, X_t)\, dt + \sigma(t, X_t)\, dW_t; \tag{SDE}$$

it remains to make sense of the **stochastic integral** in the integral form of this equation,

$$X_t = x_0 + \int_0^t \mu(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s. \tag{SIE}$$

The Brownian Bridge, our first example, had $x_0 = 0$, $\mu(s,x) = -x/(1-s)$, and $\sigma(s,x) = 1$; the Wiener process (scaled Browian motion with drift) could be written in the same form with constant $\mu(s,x) \equiv \mu$ and $\sigma(s,x) \equiv \sigma$, while standard Brownian motion itself has simply $x_0 = 0$, $\mu(s,x) \equiv 0$, and $\sigma(s,x) \equiv 1$. Our next diffusion, the Ornstein-Uhlenbeck velocity (OUV) process, is the continuous-time analogue of an AR(1) time-series, with $\mu(s,x) = -\beta x$ and $\sigma(s,x) \equiv 1$, so its SIE representation will be:

$$X_t = x_0 - \beta \int_0^t X_s\, ds + W_t.$$

Lemma (Gronwall's Inequality). *Suppose for $t > 0$ the nonnegative function $f(t) \ge 0$ satisfies the inequality*

$$f(t) \le \alpha_t + \beta \int_0^t f(s)\,ds.$$

Then $f(t) \le \alpha_t + \beta \int_0^t e^{\beta(t-s)} \alpha_s\,ds$ for all $t > 0$. In particular, if $\alpha_s \equiv \alpha$, then $f(t) \le \alpha e^{\beta t}$, and if $\alpha_s \equiv 0$, then $f(t) \le \beta \int_0^t f(s)\,ds$ entails $f(t) \equiv 0$.

Proof. *Iterate:*

$$\begin{aligned}
f(t) &\le \alpha_t + \beta \int_0^t f(s)\,ds \\
&\le \alpha_t + \beta \int_0^t \Big(\alpha_{s'} + \beta \int_0^{s'} f(s)\,ds\Big)\,ds' \\
&= \alpha_t + \beta \int_0^t (1)\,\alpha_s\,ds + \beta \int_0^t \beta(t-s) f(s)\,ds \\
&\le \alpha_t + \beta \int_0^t \alpha_s\,ds + \beta \int_0^t \beta(t-s')\Big(\alpha_{s'} + \beta \int_0^{s'} f(s)\,ds\Big)\,ds' \\
&= \alpha_t + \beta \int_0^t \big(1 + \beta(t-s)\big)\alpha_s\,ds + \beta \int_0^t \frac{\beta^2 (t-s)^2}{2!} f(s)\,ds \\
&\le \dots \le \alpha_t + \beta \int_0^t \left(1 + \dots + \frac{\beta^n (t-s)^n}{n!}\right) \alpha_s\,ds + \beta \int_0^t \frac{\beta^{(n+1)} (t-s)^{(n+1)}}{(n+1)!} f(s)\,ds \\
&\le \alpha_t + \beta \int_0^t e^{\beta(t-s)} \alpha_s\,ds.
\end{aligned}$$

□

Theorem (SDE). *Suppose for $t > 0$ and $x \in \mathbb{R}$ the function $\mu(t,x) \in \mathbb{R}$ satisfies the Lipschitz inequality and linear growth condition*

$$|\mu(s,x) - \mu(s,y)| \le \beta |x-y| \qquad |\mu(s,x)|^2 \le \beta\big(1 + |x|^2\big).$$

Then for any continuous function $w : \mathbb{R}_+ \to \mathbb{R}$ and numbers $x_0 \in \mathbb{R}$, $\sigma > 0$, there is a unique solution to the integral equation

$$X_t = x_0 + \int_0^t \mu(s, X_s)\,ds + \sigma w_t.$$

Proof. *For any continuous $f \in \mathcal{C}(\mathbb{R}_+)$ define a new element $Tf \in \mathcal{C}(\mathbb{R}_+)$ by:*

$$Tf(t) \equiv x_0 + \int_0^t mu\big(s, f(s)\big)\,ds + \sigma w_t.$$

Starting with $X^{(0)} \equiv x_0$, define a succession of approximations to X_t by

$$X^{(n)} \equiv TX^{(n-1)}(t) \equiv x_0 + \int_0^t \mu\big(s, X^{(n-1)}(s)\big)\,ds + \sigma w_t.$$

Then $f^{(n)}(t) \equiv |X^{(n+1)}(t) - X^{(n)}(t)|$ satisfies

$$\begin{aligned}
f^{(n)}(t) &\le \int_0^t |\mu(s, X^{(n)}) - \mu(s, X^{(n-1)})|\,ds \\
&\le \int_0^t \beta |X^{(n)} - X^{(n-1)}|\,ds = \beta \int_0^t f^{(n-1)}(s)\,ds,
\end{aligned}$$

so we can mimic the Gronwall proof:

$$\begin{aligned}
f^{(n)}(t) &\le \beta \int_0^t f^{(n-1)}(s)\,ds \le \beta \iint\limits_{0\le s\le s'\le t} \beta f^{(n-2)}(s)\,ds\,ds' \\
&= \beta^2 \int_0^t (t-s) f^{(n-2)}(s)\,ds \le \beta^2 \iint\limits_{0\le s\le s'\le t} (t-s')\beta f^{(n-3)}(s)\,ds\,ds' \\
&= \beta^3 \int_0^t \frac{(t-s)^2}{2!} f^{(n-3)}(s)\,ds \le \cdots \le \\
&\le \beta^n \int_0^t \frac{(t-s)^{n-1}}{(n-1)!} f^{(0)}(s)\,ds \\
&\to 0
\end{aligned}$$

as $n \to \infty$ *by Lebesgue's dominated convergence theorem, since* $\sum \frac{\beta^n (t-s)^{n-1}}{(n-1)!} = \beta e^{\beta(t-s)} < \infty$. ☐