Sta 961: Homework #1

Let $\mathcal{H}_s$ be the Hilbert space completion of $\mathcal{C}_c^\infty(0,1)$ in the inner product

$$\langle f,\, g\rangle_s = \sum_{n\in\mathbb{N}} (\pi^2 n^2)^s a_n b_n,$$

where $a_n := \int f(s)\,\psi_n(s)\,ds$ and $b_n := \int g(s)\,\psi_n(s)\,ds$ (all integrals are over $[0,1]$) for

$$\psi_n(s) := \sqrt{2}\,\sin(n\pi s)$$

and, for $0 < s,t < 1$, set

$$\gamma(s,t) := s \wedge t - st.$$

1. For which $s \in \mathbb{R}$ is $f(x) := x(1-x)$ in $\mathcal{H}_s$? Find $\|f\|_1^2$.

2. Let $B_t \sim \mathsf{No}\big(0, \gamma()\big)$ be the Brownian Bridge. Is B_s a Markov process? Compute the conditional expectations

$$\mathsf{E}[B_u \mid B_s = x, B_t = y] \quad \text{and} \quad \mathsf{E}[B_u \mid B_t = y]$$

for $0 < s < t < u < 1$ (if they differ then B_s can't be Markov). If this doesn't show that B_s is *not* Markov, try to use one of the representations of $B(t)$ in Section 2 of class lecture notes "Dirichlet Sobolev Spaes on $[0,1]$ and the Brownian Bridge" to prove that it *is*.

3. Let W be the unit Gaussian process on $\mathcal{H}_0$, *i.e.*, an isomorphism

$$W:\ \mathcal{H}_0 \to L_2(\Omega, \mathcal{F}, \mathsf{P})$$

that maps elements $\phi \in \mathcal{H}_0$ to Gaussian random variables $W[\phi] \sim \mathsf{No}\big(0, \|\phi\|_0^2\big)$ such that $\mathsf{Cov}\big(W[\phi], W[\psi]\big) = \langle \phi,\, \psi\rangle_0 = \int \phi(s)\psi(s)\,ds$. For $0 \le t \le 1$ set $g_t(s) := \mathbf{1}_{\{s\le t\}}$ and $X_t := W[g_t]$. Find the mean, variance, and covariance functions for X_t explicitly.

4. The Gaussian process Y_s on the unit interval with mean zero and covariance function

$$h(s,t) = \tfrac{1}{2}(s-t)^2 - \tfrac{1}{2}|s-t| + 1/12$$

is "periodic Brownian motion," with paths that are periodic functions on $[0,1]$. Show that the sample paths are orthogonal to the constants in $\mathcal{H}_0$— *i.e.*, that $\int Y_s\,ds = 0$ almost surely. (Hint: Set $Z := \int Y_s\,ds$ and consider $\mathsf{Cov}(Y_s, Z)$ or $\mathsf{V}(Z)$).

5. What is the generator $\mathcal{G}$ for the periodic Brownian Motion process Y_s above? Find $\mathcal{G}$ such that

$$\mathsf{E}Y[\phi]Y[\mathcal{G}\psi] = \langle \phi,\, \psi\rangle_0$$

(Hint: $\mathcal{G}$ will be the inverse of the integral operator that takes ψ to $g(s) = \int h(s,t)\psi(t)\,dt$. Try taking a derivative or two of g and try to find $\mathcal{G}$ s.t. $\mathcal{G}g = \psi$). Find the domain $\mathcal{D}(\mathcal{G})$ and the value of $\mathcal{G}g$ for $g \in \mathcal{G}(\mathcal{G})$.

Last edited: February 11, 2014