Free Sobolev Space on $\mathbb{R}^1$ and the Orstein-Uhlenbeck Process

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1 Introduction to the O-U Process

When we built the Brownian Bridge and Dirichlet Sobolev Space, we exploited the fact that BB's covariance function $\gamma_{\text{BB}}(s,t) = s \wedge t - st$ was the Greens Function for the operator $(-\Delta)$ with Dirichlet boundary conditions on $[0,1]$, *i.e.*, the formal solution there to $(-\Delta)\gamma_{\text{BB}\,s}(t) = \delta_s(t)$ or, less formally, the kernel $\gamma_{\text{BB}}(s,t)$ for which the integral operator

$$\mathfrak{G}_{\text{BB}}\psi(s) := \int_0^1 \gamma_{\text{BB}}(s,t)\,\psi(t)\,dt$$

has the property that $g(s) = \mathfrak{G}_{\text{BB}}\psi(s)$ solves

$$(-\Delta)g(s) = \psi(s)$$

on $[0,1]$ with Dirichlet boundary conditions $g(0) = 0 = g(1)$. Now we wish to build a similar Gaussian process with the entire real line $\mathbb{R}$ for parameter space. Because the negative Laplacian is only positive *semi*-definite on $\mathbb{R}$ (while the Dirichlet Laplacian was strictly positive definite), we can't quite do exactly the same thing— for an L_2 function γ_s to satisfy $(-\Delta)\gamma_s(t) = 0$ for $t \neq s$, γ_s would have to be a linear function $\gamma_s(t) = a + bt$ away from $t = s$ and the only linear function in $L_2(\mathbb{R})$ is identically zero. We can solve the problem by modifying things just a bit, by adding a positive constant β^2 to $(-\Delta)$.

The formal solution to the equation $(-\Delta + \beta^2)\gamma_s(t) = \delta_s(t)$ that lies in $L_2(\mathbb{R})$ is

$$\gamma(s,t) := \frac{1}{2\beta}\,e^{-\beta|s-t|},$$

the Greens function for $(-\Delta + \beta^2)$ on $\mathbb{R}$. The mean-zero Gaussian process X_t with covariance γ is called the "Ornstein-Uhlenbeck velocity process" or, more commonly, the "O-U process". It:

- is **Stationary** because $\mathsf{Cov}(X_s, X_t) = \gamma(s,t)$ depends only on $s - t$;
- is **Markov** because X_s and X_u are conditionally independent given X_t for any $s < t < u$;
- has **Continuous Paths** by Kolmogorov's theorem because $X_t - X_s \sim \mathsf{No}\big(0, \frac{1}{\beta}[1 - e^{-\beta|s-t|}]\big)$, so $\mathsf{E}|X_t - X_s|^2 \le |t-s|$ and $\mathsf{E}|X_t - X_s|^{2n} \le \frac{(2n)!}{2^n\,n!}|t-s|^n$. By Kolmogorov's theorem the paths are γ-Hölder continuous for any $\gamma < \frac{n-1}{2n}$, hence for any $\gamma < 1/2$.

2 Review of Fourier Transforms

Below we will need a few properties of Fourier transforms on $\mathbb{R}^1$, which we collect here.

Definition 1 (Fourier Transform). *Let $f \in \mathcal{C}_c^\infty(\mathbb{R})$. The* Fourier Transform *of f is the function*

$$\hat{f}(\xi) := \int_{\mathbb{R}} e^{is\xi} f(s)\, ds$$

When necessary for clarity we will also denote the transform by $\mathcal{F}[f] = \hat{f}$. Here we collect some properties that will prove to be useful in Section (3) below. For $f, g \in \mathcal{C}_c^\infty$, at least,

The derivative $f'(s)$ has transform

$$\mathcal{F}[f'](\xi) = (-i\xi)\, \hat{f}(\xi); \tag{1a}$$

and the kth derivative has transform

$$\mathcal{F}[f^{(k)}](\xi) = (-i\xi)^k\, \hat{f}(\xi); \tag{1b}$$

The derivative *of* $\hat{f}(\xi)$ is:

$$(\hat{f})'(\xi) = \mathcal{F}[(is)f(s)], \tag{1c}$$

the Fourier Transform of $s \mapsto (is)f(s)$, and the kth derivative is:

$$\hat{f}^{(k)}(\xi) = \mathcal{F}[(is)^k f(s)]. \tag{1d}$$

The offset $f_t(s) := f(s-t)$ has transform

$$\mathcal{F}[f_t](\xi) = e^{it\xi}\, \hat{f}(\xi) \tag{1e}$$

and the reverse $r(s) = f(-s)$ has transform

$$\hat{r}(\xi) = -\hat{f}(-\xi). \tag{1f}$$

The L_2 inner product of $\hat{f}$ and $\hat{g}$ is

$$\begin{aligned}
\int \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi &= \iiint f(s)e^{is\xi}\overline{g(t)e^{it\xi}}\, ds\, dt\, d\xi \\
&= \lim_{\epsilon\to 0} \iint f(s)\overline{g(t)} \left\{ \int e^{i(s-t)\xi - \epsilon\xi^2/2}\, d\xi \right\} ds\, dt \\
&= \lim_{\epsilon\to 0} \iint f(s)\overline{g(t)} \left\{ \sqrt{2\pi/\epsilon}\; e^{-(s-t)^2/2\epsilon} \right\} ds\, dt \\
&= 2\pi \int f(s)\overline{g(s)}\, ds
\end{aligned} \tag{1g}$$

The product of $\hat{f}$ and $\hat{g}$ is

$$\begin{aligned}
\hat{f}(\xi)\hat{g}(\xi) &= \iint f(s)e^{is\xi}\, g(t)e^{it\xi}\, ds\, dt \\
&= \iint f(s)e^{is\xi}\, g(u-s)e^{i(u-s)\xi}\, ds\, du \\
&= \int \left\{ \int f(s)\, g(u-s)\, ds \right\} e^{iu\xi}\, du \\
&= \widehat{f \star g}(\xi)
\end{aligned}$$

for the *convolution*

$$f \star g(s) := \int f(s)\, g(u-s)\, ds. \tag{1h}$$

By Lebesgue's DCT and the continuity at every $t \in \mathbb{R}$ of $f \in \mathcal{C}_c^\infty$,

$$\begin{aligned}
\int \hat{f}(\xi)e^{-it\xi}\, d\xi &= \iint f(s)e^{i(s-t)\xi}\, d\xi\, ds \\
&= \lim_{\epsilon \to 0} \int f(s) \left\{ \int e^{i(s-t)\xi - \epsilon\xi^2/2}\, d\xi \right\} ds \\
&= \lim_{\epsilon \to 0} \int f(s) \left\{ \sqrt{2\pi/\epsilon}\; e^{-(s-t)^2/2\epsilon} \right\} ds \\
&= 2\pi f(t)
\end{aligned}$$

so f can be recovered from $h = \hat{f}$ by the "inverse Fourier transform" as

$$\mathcal{F}^{-1}[h](t) = \check{h}(t) := \frac{1}{2\pi} \int e^{-it\xi} h(\xi)\, d\xi. \tag{1i}$$

3 Free Sobolev Spaces on $\mathbb{R}^1$

Denote by $\mathcal{H}_0 = L_2(\mathbb{R})$ the complex Hilbert space completion of $\mathcal{C}_c^\infty$ in the inner product

$$\langle f,\, g\rangle_0 := \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt$$

and by $\mathcal{H}_{-1}$ that with inner product

$$\langle f,\, g\rangle_{-1} := \iint_{\mathbb{R}^2} f(s)\gamma(s,t)\overline{g(t)}\, ds\, dt = \mathsf{E} X[f]\, \overline{X[g]},$$

for $X[f] := \int_{\mathbb{R}} f(s)\, X_s\, ds$ and $X[g] := \int_{\mathbb{R}} g(t)\, X_t\, dt$.

By (1g) we can evaluate $\langle f,\, g\rangle_0$ from the Fourier transforms of f and g as

$$\begin{aligned}
\langle f,\, g\rangle_0 &:= \int_{\mathbb{R}} f(t)\overline{g(t)}\, dt \\
&= \frac{1}{2\pi} \int \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi
\end{aligned}$$

Interestingly, we can also do this for $\langle f,\, g\rangle_{-1}$:

$$\begin{aligned}
\langle f,\, g\rangle_{-1} &:= \iint f(s)\gamma(s,t)\overline{g(t)}\,ds\,dt\\
&= \frac{1}{2\pi}\int \hat{f}(\xi)\overline{\mathcal{F}\Big[\int \gamma(s,t)\,g(t)\,dt\Big](\xi)}\,d\xi\\
&= \frac{1}{2\pi}\int \hat{f}(\xi)\overline{\hat{g}(\xi)}(\xi^2+\beta^2)^{-1}\,d\xi, \qquad \text{since}\\
\mathcal{F}\Big[\int \gamma(s,t)\,g(t)\,dt\Big](\xi) &= \frac{1}{2\beta}\int e^{is\xi}\left\{\int e^{-\beta|s-t|}g(t)\,dt\right\}\,ds\\
&= \frac{1}{2\beta}\int g(t)\left\{\int e^{is\xi-\beta|s-t|}\,ds\right\}\,dt\\
&= \frac{1}{2\beta}\int g(t)e^{it\xi}\left\{\int_{-\infty}^{t} e^{(s-t)(i\xi+\beta)}\,ds+\int_{t}^{\infty} e^{(s-t)(i\xi-\beta)}\,ds\right\}\,dt\\
&= \frac{1}{2\beta}\int g(t)e^{it\xi}\left\{\frac{1}{i\xi+\beta}-\frac{1}{i\xi-\beta}\right\}\,dt\\
&= \int g(t)e^{it\xi}(\xi^2+\beta^2)^{-1}\,dt\\
&= \hat{g}(\xi)(\xi^2+\beta^2)^{-1}.
\end{aligned}$$

Thus we have

$$\begin{aligned}
\langle f,\, g\rangle_0 &:= \int_{\mathbb{R}} f(t)\overline{g(t)}\,dt &&= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi\\
\langle f,\, g\rangle_{-1} &:= \iint_{\mathbb{R}^2} f(s)\gamma(s,t)\overline{g(t)}\,ds\,dt &&= \frac{1}{2\pi}\int_{\mathbb{R}} (\xi^2+\beta^2)^{-1}\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi,
\end{aligned}$$

suggesting the extension to

$$\langle f,\, g\rangle_s := \frac{1}{2\pi}\int_{\mathbb{R}} (\xi^2+\beta^2)^s\hat{f}(\xi)\overline{\hat{g}(\xi)}\,d\xi \tag{2}$$

for all $s \in \mathbb{R}$, and the Hilbert-space completion $\mathcal{H}_s$ of $\mathcal{C}_c^\infty(\mathbb{R})$ in the norm $|f|_s := \langle f,\, f\rangle_s^{1/2}$. This is called the free Sobolev space on $\mathbb{R}^1$ of index s.

From (1b) we have $\mathcal{F}[(-\Delta)f](\xi) = \xi^2\hat{f}(\xi)$, so the Fourier transform of $(-\Delta+\beta^2)f$ is $(\xi^2+\beta^2)\hat{f}$. It follows that $(-\Delta+\beta^2)$ is an isometry from any $\mathcal{H}_s$ to $\mathcal{H}_{s-2}$ for each $s \in \mathbb{R}$, and

$$\mathfrak{G}:\ \psi \mapsto \mathfrak{G}[\psi](t) := \int \gamma(s,t)\,\psi(t)\,dt$$

is an isometry from $\mathcal{H}_s$ to $\mathcal{H}_{s+2}$. Also, integration by parts gives

$$\begin{aligned}
\langle f,\, g\rangle_1 &= \int f(s)(-\Delta+\beta^2)\overline{g(s)}\,ds\\
&= -\int f(s)\,\overline{g''(s)}\,ds + \beta^2\int f(s)\overline{g(s)}\,ds\\
&= \int f'(s)\,\overline{g'(s)}\,ds + \beta^2\int f(s)\overline{g(s)}\,ds\\
&= \langle f',\, g'\rangle_0 + \beta^2\langle f,\, g\rangle_0
\end{aligned}$$

so $f \in \mathcal{H}_1$ if and only if both f and f' are in $L_2 = \mathcal{H}_0$.

4 More on O-U

The "spectrum" of an operator $\mathfrak{A}$ is the set of its eigenvalues or, more precisely, is the set $\{\lambda \in \mathbb{C} : (\mathfrak{A} - \lambda I)$ does not have a bounded inverse$\}$ (here I is the identity operator). In the Dirichlet case on the unit interval, the eigenvalues of the negative Laplacian $(-\Delta)$ were the discrete set of numbers $\{\lambda_n = (n\pi)^2 : n \in \mathbb{N}\}$, with corresponding eigenfunctions $\psi_n(s) = \sqrt{2}\sin(n\pi s)$. This was the key to our construction of the Brownian Bridge as a sum

$$B_t = \sum_{n\in\mathbb{N}} \frac{1}{\sqrt{\lambda_n}} Z_n \, \psi_n(t).$$

The spectrum of the "Free" negative Laplacian $(-\Delta)$ on $\mathbb{R}$ is *not* a discrete set— rather, it is the entire interval $[0,\infty)$, and that of $(-\Delta + \beta^2)$ is the interval $[\beta^2, \infty)$. No unit vector $\psi \in L_2$ is an exact eigenvector satisfying $(-\Delta)\psi = \lambda\psi$ for any $\psi \in \mathcal{C}$, but we can come arbitrarily close to satisfying that equation with function like

$$\psi(t) = \sin(t\sqrt{\lambda} - \epsilon t^2/2)$$

for small $\epsilon > 0$. While we can't write X_t as a discrete sum, we can write it as a Wiener stochastic integral

$$X_t = \int_{\mathbb{R}} \phi(t-u)\, dW_u$$

for any $\phi \in L_2$ for which

$$\begin{aligned}
\frac{1}{2\beta} e^{-\beta|s-t|} &= \mathsf{E}\, X_s \overline{X_t} \\
&= \int \phi(s-u)\, \overline{\phi(t-u)}\, du \\
&= \int \phi(x)\, \overline{\phi(t-s+x)}\, dx && (x = (s-u)) \\
&= \frac{1}{2\pi} \int \hat{\phi}(\xi)\ \overline{\hat{\phi}(\xi) e^{i(s-t)\xi}}\, d\xi && \text{(by (1e))} \\
&= \mathcal{F}^{-1}\big[\, |\hat{\phi}|^2 \,\big](s-t), && \text{(by (1i)), so} \\
|\hat{\phi}|^2(\xi) &= \frac{1}{2\beta} \int e^{it\xi} e^{-\beta|t|}\, dt \\
&= \frac{1}{\beta^2 + \xi^2}
\end{aligned}$$

Possible solutions include the symmetric choice $\phi = \mathcal{F}^{-1}\big[(\beta^2+\xi^2)^{-1/2}\big] = \frac{1}{\pi} K_0\big(\beta|t|\big)$ (K_0 is the modified Bessel function of the second kind), and the *causal* or *nonantipicating* choice $\phi(t) = e^{-\beta t}\mathbf{1}_{\{t>0\}}$, each with $|\hat{\phi}(\xi)|^2 = (\beta^2+\xi^2)^{-1}$.