Final Examination

Mth 136 = Sta 114

Monday, 2011 May 2, 7:00 – 10:00pm

- This is a **closed book** exam— please put your books on the floor.
- You may use a calculator and **two pages** of your own notes.
  Do not share calculators or notes.
- **Show** your **work**. Neatness counts. Boxing answers helps.
- **Simplify all expressions** for full credit. Numerical answers:
  four significant **dig**its or fractions in lowest terms.
- Distribution & pdf/pmf tables and blank worksheet are attached.

Cheating on exams is a breach of trust with classmates and faculty that will not be tolerated. After completing the exam please acknowledge that you have adhered to the Duke Community Standard:

- I will not lie, cheat, or steal in my academic endeavors;
- I will conduct myself honorably in all my endeavors; and
- I will act if the Standard is compromised.

Please affirm the Community Standard with your signature below.

Signature: ______________________

Print Name: ______________________

1.	/20	6.	/20
2.	/20	7.	/20
3.	/20	8.	/20
4.	/20	9.	/20
5.	/20	10.	/20
Total:			/200

Problem 1: Smokey Joe likes to cook outside on the grill. The thing he likes best is that, every now and then the coals burst into violent flame and totally incinerate whatever he's cooking. Each day of the week this happens independently, always with the same probability p (which might be different one week than another). It's very exciting.

Each day, the number of hamburgers Smokey cooks before the bonfire has a geometric probability distribution, with pmf

$$\mathsf{P}[X_i = x \mid p] = p(1-p)^x, \quad x = 0, 1, 2, \cdots$$

a) Find a sufficient statistic $T_n(\mathbf{x})$ for n observations $\mathbf{x} = \{x_1, ..., x_n\}$ from this distribution, and explain why T_n is sufficient.

$T_n(\mathbf{x}) =$ ____________________

b) Find the Maximum Likelihood Estimate (MLE) $\hat{p}(\mathbf{x})$ for p, if we observe the data set $\mathbf{x} = \{7, 5, 9, 6, 3\}$. Show your work— *derive* the result, don't just write down the answer. Answer is numeric.

$\hat{p} =$ ____________________

Problem 1 (cont):

c) For the same data set $\mathbf{x} = \{7, 5, 9, 6, 3\}$, with a uniform prior distribution $\pi(p) = \mathbf{1}_{\{0<p<1\}}$, find the posterior mean and variance:[1]

$\mathsf{E}[p \mid \mathbf{x}] =$ $\mathsf{Var}[p \mid \mathbf{x}] =$

d) With the same uniform prior distribution $\pi(p) = \mathbf{1}_{\{0<p<1\}}$, find the posterior probability that $p \le \frac{1}{2}$ for a different week— a really bad one with seven mishaps in a row, $\mathbf{x} = \{0, 0,0, 0, 0, 0, 0\}$.

$\mathsf{P}\left[p < \frac{1}{2} \mid \mathbf{x}\right] =$

[1] Hint: No integration is needed for part c). Answer is numeric.

Problem 2: Let X have the Poisson distribution $\mathsf{Po}(\lambda)$ with pmf

$$f(x \mid \lambda) = \frac{\lambda^x}{x!} e^{-\lambda}, \qquad x = 0, 1, 2, \cdots.$$

a) Write the pmf for X in natural exponential family form

$$f(x \mid \eta) = e^{\eta \cdot T(x) - A(\eta)}\ h(x)$$

by identifying the:

Natural Parameter $\eta(\lambda)=$

Sufficient Statistic $T(x)=$

Normalizing Constant $A(\eta)=$

Data Function $h(x)=$

b) Find the Fisher Information[2] $I_\eta(\eta)$ for the natural parameter[3]:
$I_\eta(\eta) =$

[2] In parts b) and c) of this problem you're asked to find two different Fisher information functions— one for X's distribution when parametrized by λ, and one for the same distribution parametrized by η. The subscript (λ or η) is just a label so we don't mix them up. Only one observation is used in parts a, b, c.

[3] Hint: Find $A(\eta)$ first! You'll need it!

Problem 2 (cont):

c) Find the Fisher Information $I_\lambda(\lambda)$ for the usual (mean) parameter:
$I_\lambda(\lambda) =$

d) The MLE $\hat\lambda(\mathbf{x}) = \bar{x}_n$ of the Poisson mean parameter λ for a sample $\mathbf{x} = \{X_1, ..., X_n\}$ of size n is unbiased. What does the Cramér-Raô (Information) Inequality say about its squared-error risk, $\mathsf{E}[(\hat\lambda(\mathbf{x}) - \lambda)^2 \mid \lambda]$?

e) Find the MLE $\hat\eta_n(\mathbf{x})$ of the natural parameter[4] η.
For just 1 pt: Do you expect that $\hat\eta_n$ is unbiased too? Why?
$\hat\eta_n(\mathbf{x}) =$ Unbiased? Y N

[4] The Invariance Principle makes this very easy

Problem 3: Quasimodo has k keys; exactly one of them opens the treasure chest in Notre Dame Cathedral. When a key is requested he picks one uniformly at random, so there is a $\theta = 1/k$ probability the key will work.[5] We don't know the value of the positive integer k. Esmerelda fears that there are one hundred keys (so $\theta = 0.01$). She wants to test the hypotheses

$$H_0: \; k \geq 100 \qquad vs. \qquad H_1: \; k < 100.$$

In a fixed number $n = 4$ of tries (independently, with replacement), Esmerelda had $X = 1$ success and $n - X = 3$ failures. Not bad.

a) Which possible outcomes with four tries *are* "as extreme or more so" evidence against H_0 than hers for this alternative?

b) Find the P-value for a test of H_0 against H_1 with $X = 1$ (and $n = 4$). Do you accept or reject H_0 at level $\alpha = 0.05$?

$P =$ ◯Accept ◯Reject

c) Captain Phoebus felt that the values $k = 10$ and $k = 100$ were equally likely (and that no other k was possible), before he learned of Esmerelda's experiment. Find the posterior probability:

$\pi(k = 100 \mid X = 1) =$

[5] Some may prefer to express all the dist'ns and hypotheses in terms of θ instead of k.

Problem 4: The count X comes from the Poisson distribution with pmf

$$p_\theta(k) = \mathsf{P}_\theta[X = k] = \frac{\theta^k}{k!} e^{-\theta}, \qquad k = 0, 1, 2, \ldots$$

with uncertain parameter $\theta > 0$. We wish to test the hypothesis H_0 that $\theta = 5.0$ against various alternatives. Here are the values of the pmf $p_5(k) \equiv \mathsf{P}[X = k \mid \theta = 5]$ and CDF $P_5(k) \equiv \mathsf{P}[X \le k \mid \theta = 5]$ for $0 \le k \le 10$:

k:	0	1	2	3	4	5	6	7	8	9	10
$p_5(k)$:	0.007	0.034	0.084	0.140	0.175	0.175	0.146	0.104	0.065	0.036	0.018
$P_5(k)$:	0.007	0.040	0.125	0.265	0.440	0.616	0.762	0.867	0.932	0.968	0.986

a) Consider the test of the hypotheses

$$H_0: \ \theta = 5.0 \qquad vs. \qquad H_1: \ \theta \ne 5.0$$

based on $n = 1$ observation that rejects H_0 when $X \in \mathcal{R} = \{0, 1; \quad 7, 8, 9, \ldots\}$, *i.e.*, when $X \notin \{2, 3, 4, 5, 6\}$. Find the size α of this test. Show your work (*e.g.*, tell what number(s) you used from the table above):
$\alpha =$

b) Give an expression (sum or integral) for the *power* of the test described in a) above, as a function of θ:
$\mathsf{pow}(\theta) =$

c) Find the P-value for a test of

$$H_0: \ \theta = 5.0 \qquad vs. \qquad H_1: \ \theta > 5.0$$

for a single observation of $X = 9$. Do we Accept or Reject at level $\alpha = 0.04$?
$P =$ ◯Accept ◯Reject

Problem 5: For 2.5pt each, circle "T" for True or "F" for False. No explanations are necessary.

a) T F The Beta prior distribution is conjugate for the Geometric sampling distribution.

b) T F If X has a χ^2_ν distribution then $\mathsf{Var} X = \nu$.

c) T F The estimator with the smallest Mean Square Error is always unbiased.

d) T F If $\{X_i\} \sim \mathsf{Be}(\alpha, 2)$ then $\bar{X}$ is sufficient for α.

e) T F UMP tests exist for all one-sided hypotheses.

f) T F Reject H_0 if you observe data X in the critical region $\mathcal{R}$.

g) T F The *power* function is the probability that H_0 is rejected.

h) T F If the P-value is above 0.99, *reject* at level $\alpha = 0.01$.

Problem 6: For 2.5pt each, circle "T" for True or "F" for False, or write short answers in the boxes. All questions all concern a random sample $\mathbf{x} = \{X_i\}_{1\le i\le N}$ from the $\mathsf{No}(\mu, \sigma^2)$ distribution. Unless told otherwise, μ and σ^2 are unknown. No explanations are necessary.

a) What prior distribution is conjugate for μ, if σ^2 is known??

b) For fixed sample size n, a likelihood ratio test of $H_0: \ \mu = 0$ *vs.* $H_0: \ \mu > 0$ of smaller size α will have higher power $1-\beta$. T F

c) The t test of $H_0: \ \mu = 0$ *vs.* $H_1: \ \mu > 0$ is UMP. T F

d) If $\sigma^2 = 32$, $N = 4$, $\bar{X} = 3$ with improper prior $\pi(\mu) = 1$ then what is the posterior distribution $\pi(\mu \mid \mathbf{x})$ for μ?

e) If H_0 is true, the P-value for the t test of $H_0: \ \mu = 0$ *vs.* $H_1: \ \mu > 0$ has the $\mathsf{Un}(0, 1)$ dist'n. T F

f) Use a χ^2 dist'n to test $H_0: \ \sigma^2 = 1$ against $H_1: \ \sigma^2 > 1$. T F

g) What's the dist'n of $\sum_{1\le i\le N}(X_i - \bar{X})^2$ if $N = 9$ and $\sigma^2 = 2$?

h) Use a t distribution to estimate μ if $\sigma^2 = 1$ and $N \le 10$. T F

Problem 7: Ada and Van Veen model the numbers of potholes on different segments of Rte I-85 as independent Poisson random variables with means λL_i, where L_i denotes the length (in miles) of the segment. Thus,

$$\{X_i\} \overset{\text{ind}}{\sim} \mathsf{Po}(\lambda L_i).$$

The data are independent, but not (quite) identically distributed, because the (known, non-random) lengths L_i vary. Ada and Van observe 10 pothole counts $\mathbf{x} = \{X_i\}$ on ten road segments of specified lengths $\{L_i\}$, with empirical pothole rates X_i/L_i, as follows:

	Data										Sum
Potholes	105	72	105	85	115	110	95	100	105	108	1000
Length (mi)	25	8	30	20	23	20	19	30	15	10	200
Rate (Ph/mi)	4.2	9.0	3.5	4.25	5.0	5.5	5.0	3.33	7.0	10.8	57.58

a) Find the likelihood function for λ for the data above. Simplify!

$f(\mathbf{x} \mid \lambda) =$

b) Find the MLE for these data:

$\hat{\lambda} =$

Problem 7 (cont):

c) Ada and Van choose a gamma prior distribution

$$\pi(\lambda) \sim \mathsf{Ga}(\alpha = 20,\ \beta = 5)$$

for the rate parameter λ. Find their posterior distribution (give its name and any parameter(s)) and their posterior mean, for these data:

$\pi(\lambda \mid \mathbf{x}) =$ $\mathsf{E}[\lambda \mid \mathbf{x}] =$

d) Briefly, *why* should $\hat{\lambda}(\mathbf{x})$ or $\mathsf{E}[\lambda \mid \mathbf{x}]$ be better estimates of the pothole rate than the average value ($\bar{R} = 5.758$) of the empirical pothole rates $R_i \equiv X_i/L_i$ given in the bottom row of the data chart?

Problem 8:

In a study of drug treatment for depression, subjects are given a "mood test" twice— once before, and once after, treatment with a drug (either Prozac or a placebo). Scores range from zero to twenty, with higher scores indicating a cheerier mood and less depression. We will assume scores are approximately normally distributed, with the same variance. The scores for the nine subjects in the treated (Prozac) group, along with the sample means $\bar{x}$ and sums-of-squares SSQ $= \sum(x_i - \bar{x})^2$, $\sum(y_i - \bar{y})^2$ were:

Subject: $i =$	1	2	3	4	5	6	7	8	9	Avg	SSQ
Pre-med X_i	3	0	6	7	4	3	2	1	4	3.333	40
Post-med Y_i	5	1	5	7	10	9	7	11	8	7.000	74
Change d_i	2	1	-1	0	6	6	5	10	4	3.667	98

a) To test the hypothesis of "no change" against the alternative that Prozac *improves* mood, what would be the *alternate* hypothesis?

$H_0 : \mu_x = \mu_y$ $\qquad\qquad$ $H_1 :$

b) Which test would you recommend, and *why*?

○ Paired t ○ χ^2 ○ Two-sample t ○ F ○ Normal

c) How many degrees of freedom does the test you recommend have?

$\nu =$

Problem 8 (cont):

As before, the mood test data are:

Subject: $i =$	1	2	3	4	5	6	7	8	9	Avg	SSQ
Pre-med X_i	3	0	6	7	4	3	2	1	4	3.333	40
Post-med Y_i	5	1	5	7	10	9	7	11	8	7.000	74
Change d_i	2	1	-1	0	6	6	5	10	4	3.667	98

d) Find a 90% confidence interval for the change in mood score, $\mu_y - \mu_x$:

e) (1pt) Any concerns about the assumptions of normality and equal variance?

Problem 9: Holden Caulfield and Susie Creamcheese wonder whether the color of jelly beans is related to their heat tolerance. They decide to take a scientific approach: they put each jelly bean into a micro-wave oven for 10 seconds, and recorded what happened. Here are their data:

	Blue	White	Black	total
Nothing	20	4	6	30
Melted	8	12	4	24
Flames	2	4	0	6
total:	30	20	10	60

a) Which test would you recommend?
◯ Paired t ◯ χ^2 ◯ Two-sample t ◯ F ◯ Normal

b) How many degrees of freedom does the test you recommend have?
$\nu =$

c) Perform the test— *i.e.*, specify the null and alternate hypotheses, pick a test statistic, find its value and its approximate dist'n under H_0, *etc.* Show your work. Do you accept or reject at level $\alpha = 0.05$? ◯ Acc ◯ Rej

d) (XC): Give the P-value to six correct digits. Show your work.

Problem 10: The Pareto distribution is often used to model "heavy-tailed" data like incomes or storm intensities where some observations are *much* larger than others. Appropriately scaled, it has pmf and CDF

$$f(x) = \theta x^{-\theta-1}\mathbf{1}_{\{x\ge 1\}} \qquad\qquad F(x) = \begin{cases} 0 & x \le 1 \\ 1 - x^{-\theta} & x > 1 \end{cases}$$

a) Find the Maximum Likelihood Estimator for a sample $\mathbf{x}$ of size n:
$\hat{\theta}_n(\mathbf{x}) =$

b) Is this an *exponential family*? If so, give the natural sufficient statistic $T_n(\mathbf{x})$ for a sample $\mathbf{x}$ of size n; if not, why not?
$T_n(\mathbf{x}) =$ Expo Fam? Y N

c) Find the P-value for the Likelihood Ratio Test of

$$H_0:\ \theta = 2 \qquad \textit{vs.} \qquad H_1:\ \theta = 1$$

for the single observation of $x = 20$:
$P =$

Problem 10 (cont):

d) Find the posterior mean of θ with an improper uniform prior distribution $\pi(\theta) \equiv \mathbf{1}_{\{\theta>0\}}$ for a sample of size $n = 1$ of $x = 20$ (hint: $a^b = e^{b \log a}$). Simplify; no integration is needed.
$\mathsf{E}[\theta \mid X = 20] =$

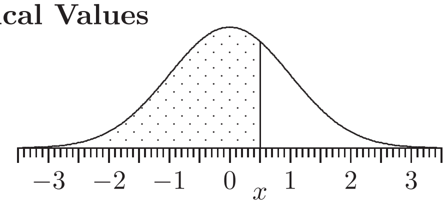
e) Show that the $\mathsf{Ga}(\alpha, \beta)$ family are conjugate prior distributions for θ. Give the parameter values for the posterior, and the *posterior mean* for a sample of size n (using your sufficient statistic T_n from part b) above):
$\pi(\theta \mid \mathbf{x}) \sim \mathsf{Ga}(\alpha^*, \beta^*)$ with:
$\alpha^* =$ $\beta^* =$ $\mathsf{E}[\theta \mid \mathbf{x}] =$

Done! Have a great summer.

(Nearly) blank worksheet, if needed:

Standard Normal CDF & Critical Values

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz:$$



Area $\Phi(x)$ under the Standard Normal Curve to the left of x.

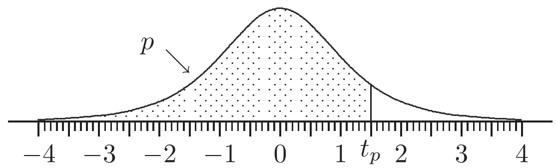
x	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
.0	.5000	.5040	.5080	.5120	.5160	.5199	.5239	.5279	.5319	.5359
.1	.5398	.5438	.5478	.5517	.5557	.5596	.5636	.5675	.5714	.5753
.2	.5793	.5832	.5871	.5910	.5948	.5987	.6026	.6064	.6103	.6141
.3	.6179	.6217	.6255	.6293	.6331	.6368	.6406	.6443	.6480	.6517
.4	.6554	.6591	.6628	.6664	.6700	.6736	.6772	.6808	.6844	.6879
.5	.6915	.6950	.6985	.7019	.7054	.7088	.7123	.7157	.7190	.7224
.6	.7257	.7291	.7324	.7357	.7389	.7422	.7454	.7486	.7517	.7549
.7	.7580	.7611	.7642	.7673	.7704	.7734	.7764	.7794	.7823	.7852
.8	.7881	.7910	.7939	.7967	.7995	.8023	.8051	.8078	.8106	.8133
.9	.8159	.8186	.8212	.8238	.8264	.8289	.8315	.8340	.8365	.8389
1.0	.8413	.8438	.8461	.8485	.8508	.8531	.8554	.8577	.8599	.8621
1.1	.8643	.8665	.8686	.8708	.8729	.8749	.8770	.8790	.8810	.8830
1.2	.8849	.8869	.8888	.8907	.8925	.8944	.8962	.8980	.8997	.9015
1.3	.9032	.9049	.9066	.9082	.9099	.9115	.9131	.9147	.9162	.9177
1.4	.9192	.9207	.9222	.9236	.9251	.9265	.9279	.9292	.9306	.9319
1.5	.9332	.9345	.9357	.9370	.9382	.9394	.9406	.9418	.9429	.9441
1.6	.9452	.9463	.9474	.9484	.9495	.9505	.9515	.9525	.9535	.9545
1.7	.9554	.9564	.9573	.9582	.9591	.9599	.9608	.9616	.9625	.9633
1.8	.9641	.9649	.9656	.9664	.9671	.9678	.9686	.9693	.9699	.9706
1.9	.9713	.9719	.9726	.9732	.9738	.9744	.9750	.9756	.9761	.9767
2.0	.9772	.9778	.9783	.9788	.9793	.9798	.9803	.9808	.9812	.9817
2.1	.9821	.9826	.9830	.9834	.9838	.9842	.9846	.9850	.9854	.9857
2.2	.9861	.9864	.9868	.9871	.9875	.9878	.9881	.9884	.9887	.9890
2.3	.9893	.9896	.9898	.9901	.9904	.9906	.9909	.9911	.9913	.9916
2.4	.9918	.9920	.9922	.9925	.9927	.9929	.9931	.9932	.9934	.9936
2.5	.9938	.9940	.9941	.9943	.9945	.9946	.9948	.9949	.9951	.9952
2.6	.9953	.9955	.9956	.9957	.9959	.9960	.9961	.9962	.9963	.9964
2.7	.9965	.9966	.9967	.9968	.9969	.9970	.9971	.9972	.9973	.9974
2.8	.9974	.9975	.9976	.9977	.9977	.9978	.9979	.9979	.9980	.9981
2.9	.9981	.9982	.9982	.9983	.9984	.9984	.9985	.9985	.9986	.9986
3.0	.9987	.9987	.9987	.9988	.9988	.9989	.9989	.9989	.9990	.9990
3.1	.9990	.9991	.9991	.9991	.9992	.9992	.9992	.9992	.9993	.9993
3.2	.9993	.9993	.9994	.9994	.9994	.9994	.9994	.9995	.9995	.9995
3.3	.9995	.9995	.9995	.9996	.9996	.9996	.9996	.9996	.9996	.9997
3.4	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9997	.9998

$\Phi(0.6745) = 0.75$ $\Phi(1.6449) = 0.95$ $\Phi(2.3263) = 0.99$ $\Phi(3.0902) = 0.999$

$\Phi(1.2816) = 0.90$ $\Phi(1.9600) = 0.975$ $\Phi(2.5758) = 0.995$ $\Phi(3.2905) = 0.9995$

Critical Values for Student's t

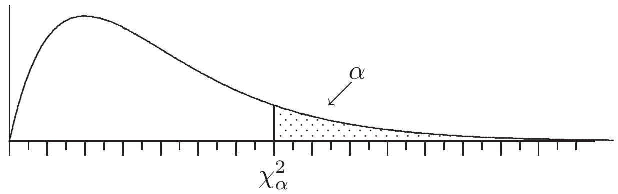
$$p = \int_{-\infty}^{t_p} c \, \frac{dt}{(1 + t^2/\nu)^{(\nu+1)/2}}$$



ν	$t_{.60}$	$t_{.70}$	$t_{.80}$	$t_{.85}$	$t_{.90}$	$t_{.95}$	$t_{.975}$	$t_{.99}$	$t_{.995}$	$t_{.999}$	$t_{.9995}$	$t_{.9999}$
1	0.325	0.727	1.376	1.9626	3.078	6.314	12.76	31.82	63.66	318.3	636.6	3183.
2	0.289	0.617	1.061	1.3862	1.886	2.920	4.303	6.965	9.925	22.33	31.60	70.70
3	0.277	0.584	0.978	1.2498	1.638	2.353	3.182	4.541	5.841	10.22	12.92	22.20
4	0.271	0.569	0.941	1.1896	1.533	2.132	2.776	3.747	4.604	7.173	8.610	13.03
5	0.267	0.559	0.920	1.1558	1.476	2.015	2.571	3.365	4.032	5.893	6.869	9.678
6	0.265	0.553	0.906	1.1342	1.440	1.943	2.447	3.143	3.707	5.208	5.959	8.025
7	0.263	0.549	0.896	1.1192	1.415	1.895	2.365	2.998	3.499	4.785	5.408	7.063
8	0.262	0.546	0.889	1.1081	1.397	1.860	2.306	2.896	3.355	4.501	5.041	6.442
9	0.261	0.543	0.883	1.0997	1.383	1.833	2.262	2.821	3.250	4.297	4.781	6.010
10	0.260	0.542	0.879	1.0931	1.372	1.812	2.228	2.764	3.169	4.144	4.587	5.694
11	0.260	0.540	0.876	1.0877	1.363	1.796	2.201	2.718	3.106	4.025	4.437	5.453
12	0.259	0.539	0.873	1.0832	1.356	1.782	2.179	2.681	3.055	3.930	4.318	5.263
13	0.259	0.538	0.870	1.0795	1.350	1.771	2.160	2.650	3.012	3.852	4.221	5.111
14	0.258	0.537	0.868	1.0763	1.345	1.761	2.145	2.624	2.977	3.787	4.140	4.985
15	0.258	0.536	0.866	1.0735	1.341	1.753	2.131	2.602	2.947	3.733	4.073	4.880
16	0.258	0.535	0.865	1.0711	1.337	1.746	2.120	2.583	2.921	3.686	4.015	4.791
17	0.257	0.534	0.863	1.0690	1.333	1.740	2.110	2.567	2.898	3.646	3.965	4.714
18	0.257	0.534	0.862	1.0672	1.330	1.734	2.101	2.552	2.878	3.610	3.922	4.648
19	0.257	0.533	0.861	1.0655	1.328	1.729	2.093	2.539	2.861	3.579	3.883	4.590
20	0.257	0.533	0.860	1.0640	1.325	1.725	2.086	2.528	2.845	3.552	3.85	4.539
21	0.257	0.532	0.859	1.0627	1.323	1.721	2.080	2.518	2.831	3.527	3.819	4.493
22	0.256	0.532	0.858	1.0614	1.321	1.717	2.074	2.508	2.819	3.505	3.792	4.452
23	0.256	0.532	0.858	1.0603	1.319	1.714	2.069	2.500	2.807	3.485	3.768	4.415
24	0.256	0.531	0.857	1.0593	1.318	1.711	2.064	2.492	2.797	3.467	3.745	4.382
25	0.256	0.531	0.856	1.0584	1.316	1.708	2.060	2.485	2.787	3.450	3.725	4.352
26	0.256	0.531	0.856	1.0575	1.315	1.706	2.056	2.479	2.779	3.435	3.707	4.324
27	0.256	0.531	0.855	1.0567	1.314	1.703	2.052	2.473	2.771	3.421	3.690	4.299
28	0.256	0.530	0.855	1.0560	1.313	1.701	2.048	2.467	2.763	3.408	3.674	4.275
29	0.256	0.530	0.854	1.0553	1.311	1.699	2.045	2.462	2.756	3.396	3.659	4.254
30	0.256	0.530	0.854	1.0547	1.310	1.697	2.042	2.457	2.750	3.385	3.646	4.234
40	0.255	0.529	0.851	1.0500	1.303	1.684	2.021	2.423	2.704	3.307	3.551	4.094
60	0.254	0.527	0.848	1.0455	1.296	1.671	2.000	2.390	2.660	3.232	3.460	3.962
120	0.254	0.526	0.845	1.0409	1.289	1.658	1.980	2.358	2.617	3.160	3.373	3.837
∞	0.253	0.524	0.842	1.0364	1.282	1.645	1.960	2.326	2.576	3.090	3.291	3.719

Critical Values for χ^2

$$\alpha = \int_{\chi^2_\alpha}^{\infty} c\, x^{\nu/2-1} e^{-x/2}\, dx$$



ν	$\chi^2_{.50}$	$\chi^2_{.25}$	$\chi^2_{.10}$	$\chi^2_{.05}$	$\chi^2_{.025}$	$\chi^2_{.01}$	$\chi^2_{.005}$	$\chi^2_{.001}$	$\chi^2_{.0005}$	$\chi^2_{.0001}$
1	0.4549	1.3233	2.7055	3.8415	5.0239	6.6349	7.87940	10.8276	12.1157	15.1367
2	1.3863	2.7726	4.6052	5.9915	7.3778	9.2103	10.5966	13.8155	15.2018	18.4207
3	2.3660	4.1083	6.2514	7.8147	9.3484	11.3449	12.8382	16.2662	17.7300	21.1075
4	3.3567	5.3853	7.7794	9.4877	11.1433	13.2767	14.8603	18.4668	19.9974	23.5127
5	4.3515	6.6257	9.2364	11.0705	12.8325	15.0863	16.7496	20.5150	22.1053	25.7448
6	5.3481	7.8408	10.6446	12.5916	14.4494	16.8119	18.5476	22.4577	24.1028	27.8563
7	6.3458	9.0371	12.0170	14.0671	16.0128	18.4753	20.2777	24.3219	26.0178	29.8775
8	7.3441	10.219	13.3616	15.5073	17.5345	20.0902	21.9550	26.1245	27.8680	31.8276
9	8.3428	11.389	14.6837	16.9190	19.0228	21.6660	23.5894	27.8772	29.6658	33.7199
10	9.3418	12.549	15.9872	18.3070	20.4831	23.2092	25.1882	29.5883	31.4198	35.5640
11	10.341	13.701	17.2750	19.6751	21.9200	24.7249	26.7568	31.2641	33.1366	37.3670
12	11.340	14.845	18.5493	21.0260	23.3366	26.2169	28.2995	32.9095	34.8213	39.1344
13	12.340	15.984	19.8119	22.3620	24.7356	27.6882	29.8195	34.5282	36.4778	40.8707
14	13.339	17.117	21.0641	23.6848	26.1189	29.1412	31.3193	36.1233	38.1094	42.5793
15	14.339	18.245	22.3071	24.9958	27.4884	30.5779	32.8013	37.6973	39.7188	44.2632
16	15.338	19.369	23.5418	26.2962	28.8453	31.9999	34.2672	39.2524	41.3081	45.9249
17	16.338	20.489	24.7690	27.5871	30.1910	33.4087	35.7185	40.7902	42.8792	47.5664
18	17.338	21.605	25.9894	28.8693	31.5264	34.8053	37.1565	42.3124	44.4338	49.1894
19	18.338	22.718	27.2036	30.1435	32.8523	36.1909	38.5823	43.8202	45.9731	50.7955
20	19.337	23.828	28.4120	31.4104	34.1696	37.5662	39.9968	45.3147	47.4985	52.3860
21	20.337	24.935	29.6151	32.6706	35.4789	38.9322	41.4011	46.7970	49.0108	53.9620
22	21.337	26.039	30.8133	33.9244	36.7807	40.2894	42.7957	48.2679	50.5111	55.5246
23	22.337	27.141	32.0069	35.1725	38.0756	41.6384	44.1813	49.7282	52.0002	57.0746
24	23.337	28.241	33.1962	36.4150	39.3641	42.9798	45.5585	51.1786	53.4788	58.6130
25	24.337	29.339	34.3816	37.6525	40.6465	44.3141	46.9279	52.6197	54.9475	60.1403
26	25.336	30.435	35.5632	38.8851	41.9232	45.6417	48.2899	54.0520	56.4069	61.6573
27	26.336	31.528	36.7412	40.1133	43.1945	46.9629	49.6449	55.4760	57.8576	63.1645
28	27.336	32.620	37.9159	41.3371	44.4608	48.2782	50.9934	56.8923	59.3000	64.6624
29	28.336	33.711	39.0875	42.5570	45.7223	49.5879	52.3356	58.3012	60.7346	66.1517
30	29.336	34.800	40.2560	43.7730	46.9792	50.8922	53.6720	59.7031	62.1619	67.6326
40	39.336	45.616	51.8051	55.7585	59.3417	63.6907	66.7660	73.4020	76.0946	82.0623
50	49.335	56.334	63.1671	67.5048	71.4202	76.1539	79.4900	86.6608	89.5605	95.9687
60	59.335	66.981	74.3970	79.0819	83.2977	88.3794	91.9517	99.6072	102.695	109.503
70	69.335	77.577	85.5270	90.5312	95.0232	100.425	104.215	112.317	115.578	122.755
80	79.334	88.130	96.5782	101.879	106.629	112.329	116.321	124.839	128.261	135.782
90	89.334	98.650	107.565	113.145	118.136	124.116	128.299	137.208	140.782	148.627
100	99.334	109.14	118.498	124.342	129.561	135.807	140.169	149.449	153.167	161.319

Name	Notation	pdf/pmf	Range	Mean μ	Variance σ^2	
Beta	$\mathsf{Be}(\alpha,\beta)$	$f(x)=\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$	$x\in(0,1)$	$\frac{\alpha}{\alpha+\beta}$	$\frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$	
Binomial	$\mathsf{Bi}(n,p)$	$f(x)=\binom{n}{x}p^x q^{(n-x)}$	$x\in 0,\cdots,n$	$n\,p$	$n\,p\,q$	$(q=1-p)$
Exponential	$\mathsf{Ex}(\lambda)$	$f(x)=\lambda\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	$1/\lambda$	$1/\lambda^2$	
Gamma	$\mathsf{Ga}(\alpha,\lambda)$	$f(x)=\frac{\lambda^\alpha}{\Gamma(\alpha)}x^{\alpha-1}\,e^{-\lambda x}$	$x\in\mathbb{R}_+$	α/λ	α/λ^2	
Geometric	$\mathsf{Ge}(p)$	$f(x)=p\,q^x$	$x\in\mathbb{Z}_+$	q/p	q/p^2	$(q=1-p)$
		$f(y)=p\,q^{y-1}$	$y\in\{1,...\}$	$1/p$	q/p^2	$(y=x+1)$
HyperGeo.	$\mathsf{HG}(n,A,B)$	$f(x)=\frac{\binom{A}{x}\binom{B}{n-x}}{\binom{A+B}{n}}$	$x\in 0,\cdots,n$	$n\,P$	$n\,P\,(1-P)\frac{N-n}{N-1}$	$(P=\frac{A}{A+B})$
Logistic	$\mathsf{Lo}(\mu,\beta)$	$f(x)=\frac{e^{-(x-\mu)/\beta}}{\beta[1+e^{-(x-\mu)/\beta}]^2}$	$x\in\mathbb{R}$	μ	$\pi^2\beta^2/3$	
Log Normal	$\mathsf{LN}(\mu,\sigma^2)$	$f(x)=\frac{1}{x\sqrt{2\pi\sigma^2}}e^{-(\log x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}_+$	$e^{\mu+\sigma^2/2}$	$e^{2\mu+\sigma^2}\left(e^{\sigma^2}-1\right)$	
Neg. Binom.	$\mathsf{NB}(\alpha,p)$	$f(x)=\binom{x+\alpha-1}{x}p^\alpha\,q^x$	$x\in\mathbb{Z}_+$	$\alpha q/p$	$\alpha q/p^2$	$(q=1-p)$
		$f(y)=\binom{y-1}{y-\alpha}p^\alpha\,q^{y-\alpha}$	$y\in\{\alpha,...\}$	α/p	$\alpha q/p^2$	$(y=x+\alpha)$
Normal	$\mathsf{No}(\mu,\sigma^2)$	$f(x)=\frac{1}{\sqrt{2\pi\sigma^2}}e^{-(x-\mu)^2/2\sigma^2}$	$x\in\mathbb{R}$	μ	σ^2	
Pareto	$\mathsf{Pa}(\alpha,\epsilon)$	$f(x)=\alpha\,\epsilon^\alpha/x^{\alpha+1}$	$x\in(\epsilon,\infty)$	$\frac{\epsilon\,\alpha}{\alpha-1}$	$\frac{\epsilon^2\alpha}{(\alpha-1)^2(\alpha-2)}$	
Poisson	$\mathsf{Po}(\lambda)$	$f(x)=\frac{\lambda^x}{x!}e^{-\lambda}$	$x\in\mathbb{Z}_+$	λ	λ	
Snedecor F	$F(\nu_1,\nu_2)$	$f(x)=\frac{\Gamma(\frac{\nu_1+\nu_2}{2})(\nu_1/\nu_2)^{\nu_1/2}}{\Gamma(\frac{\nu_1}{2})\Gamma(\frac{\nu_2}{2})}\times$ $x^{\frac{\nu_1-2}{2}}\left[1+\frac{\nu_1}{\nu_2}\,x\right]^{-\frac{\nu_1+\nu_2}{2}}$	$x\in\mathbb{R}_+$	$\frac{\nu_2}{\nu_2-2}$	$\left(\frac{\nu_2}{\nu_2-2}\right)^2\frac{2(\nu_1+\nu_2-2)}{\nu_1(\nu_2-4)}$	
Student t	$t(\nu)$	$f(x)=\frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}}[1+x^2/\nu]^{-(\nu+1)/2}$	$x\in\mathbb{R}$	0	$\nu/(\nu-2)$	
Uniform	$\mathsf{Un}(a,b)$	$f(x)=\frac{1}{b-a}$	$x\in(a,b)$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$	
Weibull	$\mathsf{We}(\alpha,\beta)$	$f(x)=\alpha\beta\,x^{\alpha-1}\,e^{-\beta\,x^\alpha}$	$x\in\mathbb{R}_+$	$\frac{\Gamma(1+\alpha^{-1})}{\beta^{1/\alpha}}$	$\frac{\Gamma(1+2/\alpha)-\Gamma^2(1+1/\alpha)}{\beta^{2/\alpha}}$	