Sta 250 = Mth 342 : Homework 4

In each of the exercises below where it appears, the symbol $\mathbf{x}$ denotes a simple random sample $\mathbf{x} = \{X_1, \ldots, X_n\}$ of n independent draws from the specified distribution.

1. Suppose that waiting times (in years, rounded up) for rare events are independent random variables with pf

$$\Pr[X_j = x] = \begin{cases} pq^{x-1} & x = 1, 2, \ldots \\ 0 & \text{other } x \end{cases}$$

   and that we observe $n = 4$ events with waiting times

$$\mathbf{x} = \{\ 10,\ 20,\ 40,\ 50\ \}$$

   Find the Maximum Likelihood Estimators for

   (a) p, the annual probability of the event; and

   (b) $\theta = 1/p$, the return rate

2. Suppose that $\mathbf{x}$ is a random sample of size n from the $\mathsf{Be}(\theta, 1)$ distribution with pdf

$$f(x \mid \theta) = \theta\, x^{\theta-1}\, \mathbf{1}_{\{0<x<1\}}$$

   Find the MLE for θ.

3. Suppose that $\mathbf{x}$ is a random sample of size n from the Laplace distribution with pdf

$$f(x \mid \theta) = \tfrac{1}{2}\, e^{-|x-\theta|}$$

   Find the MLE for θ.

4. Let α be the population median of the $\mathsf{Be}(\theta, 1)$ distribution (recall Exercise (2)), *i.e.*, the number such that $\Pr[X \le \alpha \mid \theta] = \frac{1}{2}$. Find the MLE $\hat{\alpha}$ for a random sample $\mathbf{x} = \{X_1, \ldots, X_n\}$ from this distribution.

5. Let $\{X_j\}$ be an infinite sequence of independent random variables from the uniform distribution $\mathsf{Un}(0, \theta)$ on the interval $[0, \theta]$. Show that the MLE $\hat{\theta}_n$ based on the random sample consisting of the first n observations is a *consistent* sequence of estimators in the sense that, for any $\epsilon > 0$,

$$\Pr[|\hat{\theta}_n - \theta| \le \epsilon] \to 1 \text{ as } n \to \infty$$

6. Suppose that $\mathbf{x}$ is a random sample of size n from the Normal $\mathsf{No}(\mu, \sigma^2)$ distribution with both mean μ and variance σ^2 unknown. Find the MLE for

   (a) The 90^{th} percentile θ, *i.e.*, the number such that $\mathsf{P}[X_j \leq \theta] = 0.90$;

   (b) The probability $\nu = \Pr[X > 2]$.

7. Let $\mathbf{x}$ be a random sample of size n from the $\mathsf{Ga}(\alpha, \beta)$ distribution with α known. Show that $S = \sum_{j=1}^{n} X_j$ is sufficient for β.

8. Let $\mathbf{x}$ be a random sample of size n from the $\mathsf{Ga}(\alpha, \beta)$ distribution with β known. Show that $T = \sum_{j=1}^{n} \log X_j$ is sufficient for α.

9. Let θ be a real-valued parameter taking values in an interval Θ (possibly unbounded) and let $\mathbf{x}$ have pdf or pf $f(\mathbf{x} \mid \theta)$, conditional on θ. Let $T = t(\mathbf{x})$ be a sufficient statistic. Show that for every prior distribution $\pi(\theta)$, the posterior distribution $\pi(\theta \mid \mathbf{x})$ of θ given $X = \mathbf{x}$ depends on $\mathbf{x}$ *only* through the value $t = t(\mathbf{x})$.