STA 250/MTH 342 Intro to Mathematical Statistics
Lab Session 9 / March 30, 2015 / Handout

In this lab session we review and implement some hypothesis tests.

See: `https://stat.duke.edu/courses/Spring15/sta250/labs/` for links to source code and data. Submit lab solutions via email to: `sta250@stat.duke.edu`. Any plots should be included in postscript form as attachments. The email subject must be "STA250 …" with "…" replaced by your name.

1: One-sample t-test. Suppose we have independent data $X_1, \cdots, X_n$ from $\mathsf{No}(\mu, \sigma^2)$, and the variance σ^2 is unknown. We do the following test,

$$H_0 : \mu = \mu_0 \quad \text{vs.} \quad H_1 : \mu \neq \mu_0.$$

In the lecture notes we defined

$$T = \frac{\sqrt{n}(\bar{X}_n - \mu_0)}{s} \sim t_{n-1},$$

and the corresponding rejection region

$$\mathscr{R}(C) = \{|T| > C\}.$$

Now we to the test in **R**, and we set $\mu_0 = 0$.

```
> base <- "https://stat.duke.edu/courses/Spring15/sta250/labs/lab9";
> download.file(paste(base,"data1.Rdata",sep="/"),"data1.Rdata","wget");
> load("data1.Rdata");
> data1;
 [1]  0.7787533  0.9435898 -0.1453734 -0.6031512  0.8058303 -2.4178814
 [7]  0.8380311 -0.3673249 -0.3779325 -1.2226983 -0.1546096  2.4170250
[13]  0.4929351  0.6918879 -0.9982076  1.0864768 -1.1788555 -0.4409982
[19]  1.4905478  0.3973025
> conf.level <- 0.95;
> alter <- c("two.sided", "less", "greater")[1];
> t.test(data1, alternative = alter, conf.level = conf.level);

        One Sample t-test

data:  data1
t = 0.4103, df = 19, p-value = 0.6861
alternative hypothesis: true mean is not equal to 0
95 percent confidence interval:
 -0.4173234  0.6208581
sample estimates:
mean of x
0.1017674
```

In the test report generated by **R**, "`t`" is our statistic T. The variable "`df`" is the degree of freedom. In this case $n = 20$ so `df = 19`.

The "`p-value`" is the p-value of the statistic, defined as the smallest α level at which observing $\mathbf{X}$ will lead to a rejection of the test. So, *if **R** reports a "`p-value`" smaller than the test level α, we should* *reject the null hypothesis.* In this example, we do not reject the null hypothesis since the p-value is greater than $\alpha = 0.05$.

The "`conf.level`" is the confidence level $1 - \alpha$, where α is the level of the test. The confidence interval is the one for μ if we forget about the hypothesis testing task.

$$\text{Confidence Interval} = \left[\bar{X}_n + \frac{F_{t_{19}}^{-1}(\frac{\alpha}{2})}{\sqrt{n}},\ \bar{X}_n + \frac{F_{t_{19}}^{-1}(1 - \frac{\alpha}{2})}{\sqrt{n}}\right].$$

Therefore one may also use the `t.test` function to find confidence interval.

This test is illustrated in Figure 1.

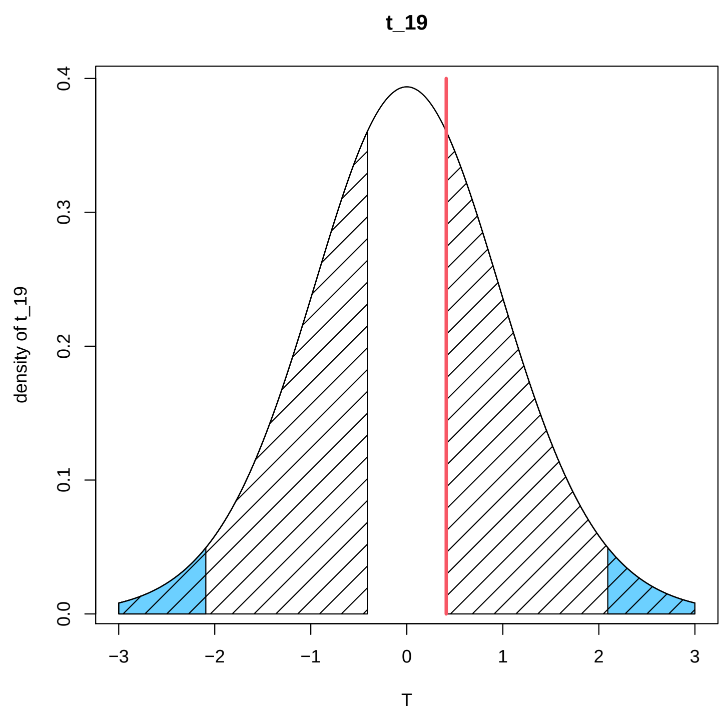

Figure 1: The two-sided t-test with level $\alpha = 0.05$. The blue part marks the rejection region of area α. The red line marks T. The area of the shaded part is the p-value.

For plotting the power function, recall that for any $\mu_1 \neq \mu_0$, and any $0 < \sigma < \infty$,

$$\pi(\mu_1, \sigma) = \Pr(\mathscr{R}(F_{t_{19}}^{-1}\left(1 - \frac{\alpha}{2}\right))|\mu_1, \sigma) = F_{t_{19}(\psi)}\left(F_{t_{19}}^{-1}\left(\frac{\alpha}{2}\right)\right) + 1 - F_{t_{19}(\psi)}\left(F_{t_{19}}^{-1}\left(1 - \frac{\alpha}{2}\right)\right),$$

where $\psi = (\mu_1 - \mu_0)\sqrt{n}/\sigma$. Below is the code for plotting this power function, and the plot is in Figure 2.

```
alpha <- 0.05
C <- qt(1-alpha/2, df = 19)
power.fun <- function(mu,sigma){
  return(pt(-C, df = 19, ncp = (mu-0)*sqrt(20)/sigma) + 1 -
    pt(C, df = 19, ncp = (mu-0)*sqrt(20)/sigma))
}
```

```
mu <- seq(-3,3,length.out = 30)
sigma <- seq(0.1, 5, length.out = 30)
z <- outer(mu, sigma, Vectorize(power.fun))
surface3d(mu, sigma, z)
# require(rgl)
# persp(mu, sigma, z, theta = -50, phi=30)
```

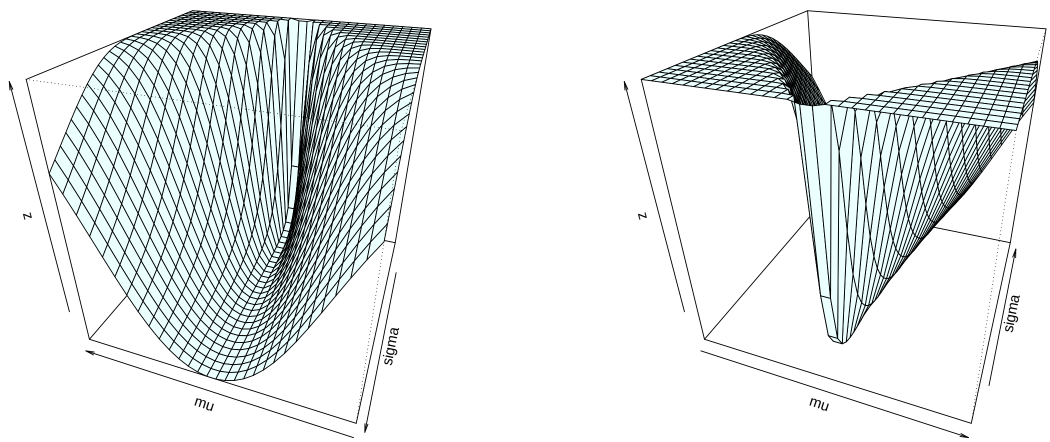

Figure 2: The power function of the above test. μ_1 runs from -3 to 3 and σ runs from 0.1 to 5.

TASK 1 With same data please do t-test with the one-sided alternative

$$H_0 : \mu \leq \mu_0 \quad \text{vs.} \quad H_1 : \mu > \mu_0,$$

where please set $\mu_0 = 0$. This could be done use the "`alternative = "greater"`" argument in the "`t.test`" function. Please also plot its power function. You may need page 21, in lecture note 17 for reference.

2: Two-sample t-test, with common variance. Now suppose we have

$$X_1, \cdots, X_n \sim \mathsf{No}(\mu_1, \sigma^2),$$
$$Y_1, \cdots, Y_m \sim \mathsf{No}(\mu_2, \sigma^2).$$

Consider the following test

$$H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 \neq \mu_2.$$

Recall the test statistic

$$T = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{(n-1)s_X^2 + (m-1)s_Y^2}{n+m-2}} \cdot \sqrt{\frac{1}{m} + \frac{1}{n}}} \sim t_{n+m-2}.$$

Recall the rejection region for level α,

$$\mathscr{R}(C) = \{|T| > C\}, \quad C = F^{-1}_{t_{n+m-2}}\left(1 - \frac{\alpha}{2}\right).$$

Let's do the test in **R**. We use $\alpha = 0.05$.

```
> download.file(paste(base,"data2.Rdata",sep="/"),"data2.Rdata","wget");
> load("data2.Rdata");
> conf.level <- 0.95;
> alter <- c("two.sided", "less", "greater")[1];
> t.test(X, Y, alternative = alter, conf.level = conf.level, var.equal = T);

        Two Sample t-test

data:  X and Y
t = -3.2534, df = 53, p-value = 0.001987
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -2.0691000 -0.4908667
sample estimates:
mean of x mean of y
0.0869087 1.3668920
```

Since "`p-value = 0.001987`", which is less than α, the null hypothesis is rejected.

TASK 2 Use the above data, test the hypothesis

$$H_0 : \mu_1 \leq \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 > \mu_2.$$

3: Welch's approximate t-test. Now suppose we have

$$X_1, \cdots, X_n \sim \mathsf{No}(\mu_1, \sigma_1^2),$$
$$Y_1, \cdots, Y_m \sim \mathsf{No}(\mu_2, \sigma_2^2).$$

Consider the following test

$$H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 \neq \mu_2.$$

Recall the Welch's approximate t-test, where a t-statistic (remember that this is just an approximation, this is strictly speaking not a t-statistic!)

$$T_w = \frac{\bar{X} - \bar{Y}}{\sqrt{\frac{s_X^2}{n} + \frac{s_Y^2}{m}}}.$$

The exact sampling distribution of T_w under H_0 is not known but it is very close to a t distribution with the following degrees of freedom,

$$\mathsf{df} = \frac{(s_X^2/n + s_Y^2/m)^2}{\frac{(s_X^2/n)^2}{n-1} + \frac{(s_Y^2/m)^2}{m-1}}.$$

We now do the test, just using the above data.

```
> load("data2.Rdata");
> conf.level <- 0.95;
> alter <- c("two.sided", "less", "greater")[1];
> t.test(X, Y, alternative = alter, conf.level = conf.level, var.equal = F);

        Welch Two Sample t-test

data:  X and Y
t = -3.3755, df = 52.949, p-value = 0.001386
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -2.040572 -0.519395
sample estimates:
mean of x mean of y
0.0869087 1.3668920
```

We see that the degree of freedom is no longer an integer. Since the p-value is small, the null hypothesis is rejected.

TASK 3 Use the above data, test the hypothesis

$$H_0 : \mu_1 \leq \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 > \mu_2.$$

4: Analyzing paired data with one-sample t-test. Suppose $X_1, \cdots, X_n$ are the numbers of hours of sleep that n individuals get on Day 1. Suppose that after some treatment, we measure the hours of sleep they get on Day 2: $Y_1, \cdots, Y_n$. Recall that in the lecture notes we treat this kind of data as paired samples, where for each i, X_i may not be independent with Y_i. We consider the hypothesis

$$H_0 : \mu_1 = \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 \neq \mu_2.$$

```
> download.file(paste(base,"data1.Rdata",sep="/"),"data1.Rdata","wget");
> load("data1.Rdata");
> conf.level <- 0.95;
> alter <- c("two.sided", "less", "greater")[1];
> t.test(X, Y, alternative = alter, conf.level = conf.level,
+   var.equal = T, paired = T);

        Paired t-test

data:  X and Y
t = -0.7304, df = 22, p-value = 0.4729
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
 -1.3397268  0.6418685
sample estimates:
mean of the differences
             -0.3489292
```

Since the p-value is greater than α, we do not reject the null hypothesis.

TASK 4 Using the above data, do the test

$$H_0 : \mu_1 \geq \mu_2 \quad \text{vs.} \quad H_1 : \mu_1 < \mu_2.$$

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