Sta 532: Homework #6

Several of these problems involve the following random sample from some probability distribution:

\[ \{X_i\} = \{1, 16, 13, 9, 30, 6, 2, 21, 1\} \quad (\ast) \]

1. Let \( \{X_i\} \sim \text{iid } \text{Un}(0, \theta) \) with \( \theta > 0 \) unknown. Based on the sample (\ast) find a symmetric 90% Confidence Interval for \( \theta \). Suggestion: First, find a pivotal quantity \( T(x) \) based on a sufficient statistic.

2. Let \( \{X_i\} \sim \text{iid } \text{Un}(0, \theta) \) with \( \theta > 0 \) unknown. Using an improper uniform prior density \( \pi(\theta) = 1_{\{\theta > 0\}} \), find a symmetric 90% Credible Interval for \( \theta \) for sample (\ast).

3. Let \( \{X_i\} \sim \text{iid } \text{Po}(\theta) \) be Poisson variables with mean \( \theta > 0 \). Find the Jeffreys’ Rule prior density \( \pi_J(\theta) \) for \( \theta \) and, based on the sample (\ast), find a symmetric 90% credible interval for \( \theta \).

4. For normal data \( \{Y_i\} \sim \text{iid } \text{No}(\mu, 1) \) with unit variance, how large a sample size \( n \) is required for a 99% Confidence Interval \([L(y), R(y)]\) for \( \mu \) to have length \(|R(y) - L(y)| < 0.01\)?

5. Let \( \{X_i\} \sim \text{iid } \text{No}(\mu, \sigma^2) \) be normally-distributed with uncertain mean and variance. Based on the sample (\ast) find a symmetric 90% Confidence Interval for \( \sigma \).

6. Find the Jeffreys’ prior \( \pi_J(\theta) \) for the Geometric distribution with pmf

\[ P_\theta[Y = y] = \theta(1 - \theta)^y, \ y \in \mathbb{Z}_+ = \{0, 1, 2, ...\} \]

for some \( \theta \in \Theta = (0, 1) \), by giving its name and the value(s) of any parameter(s). Also find the posterior distribution \( \pi_J(\theta \mid y) \) for a sample \( y = \{Y_1, \cdots, Y_n\} \), the posterior mean \( \mathbb{E}_J[\theta \mid y] \), and the MLE \( \hat{\theta}(y) \).

7. In fact the data in (\ast) were generated from a geometric distribution. Find an objective Bayes 90% Credible Interval for \( \theta \) from these data under the model \( \{X_i\} \sim \text{iid } \text{Ge}(\theta) \).
8. Find a 90% Confidence Interval for $\theta$ based on the sample $(*)$, for the model $\{X_i\} \overset{iid}{\sim} \text{Ge}(\theta)$.

Suggestion: You will need to know (or figure out) the probability distribution for the natural sufficient statistic $T(\mathbf{x})$ for a sample of size $n$ from $\text{Ge}(\theta)$, and recall from course notes “ci.pdf” that for integer-valued distributions with CDF $F_\theta(x) = P_\theta[X \leq x]$, a symmetric 100$\gamma$% confidence interval $[L(X), R(X)]$ can be found by evaluating

- If $F_\theta(x)$ is monotone decreasing in $\theta$ for each fixed $x$,

$$L(x) := \sup \{ \theta : F_\theta(x - 1) \geq \frac{1 + \gamma}{2} \} \quad R(x) := \inf \{ \theta : F_\theta(x) \leq \frac{1 - \gamma}{2} \}$$

- If $F_\theta(x)$ is monotone increasing in $\theta$ for each fixed $x$,

$$L(x) := \inf \{ \theta : F_\theta(x) \leq \frac{1 - \gamma}{2} \} \quad R(x) := \sup \{ \theta : F_\theta(x - 1) \geq \frac{1 + \gamma}{2} \}$$

any of which can be found using R.