1. For $p_0 \in (0, 1)$ and $n \in \mathbb{N}$, describe in detail the generalized likelihood ratio test (GLRT) of $H_0 : p = p_0$ against the two-sided alternative $H_1 : p \neq p_0$ for a single observation of $X \overset{iid}{\sim} \text{Bi}(n, p)$, of any size $\alpha$. Precisely what is the rejection region $R$?

2. For $p_0 \in (0, 1)$ and $n \in \mathbb{N}$, describe in detail the GLRT of the one-sided hypothesis $H_0 : p \geq p_0$ against $H_1 : p < p_0$ for a single observation of $X \overset{iid}{\sim} \text{Bi}(n, p)$, of size $\alpha$. Precisely what is the rejection region $R$?

3. For $\sigma^2_0 > 0$ and $n \in \mathbb{N}$, describe in detail the GLRT of $H_0 : \sigma^2 = \sigma^2_0$ against $H_1 : \sigma^2 \neq \sigma^2_0$ for a sample of size $n$ of $\{X_i\} \overset{iid}{\sim} \text{No}(\mu, \sigma^2)$, with unknown $\mu$, of size $\alpha$. Precisely what is the rejection region $R$?

4. It is generally believed that long-life light bulbs last no more than twice as long as standard light bulbs. In an experiment, the lifetimes $X$ of a standard bulb and $Y$ of a long-life bulb were recorded as $X = 1, Y = 5$. Modeling the lifetimes as exponential random variables of rates $\lambda$ and $\theta$ (hence means $1/\lambda$ and $1/\theta$), respectively, construct (i) a 90% confidence interval for the ratio $\theta/\lambda$, and (ii) a test of size $\alpha = 0.05$ of the hypothesis $\lambda < 2\theta$. Suggestion: Find a pivotal quantity related to $\theta/\lambda$.

5. A local councilor suspects that traffic conditions have become more hazardous in Ambridge than in Borchester, so she records the numbers $A$ and $B$ of accidents in each place in the course of a month. Assuming that $A$ and $B$ are independent Poisson random variables with rates $\lambda$ and $\theta$, it is desired to construct a test of size $\alpha \approx 1/16$ of $H_0 : \lambda \geq \theta$ against $H_1 : \lambda < \theta$. Show that:

   (a) $A + B$ is Poisson with rate $(\lambda + \theta)$;
   (b) Conditional on the value $n$ of $A + B$, that $A \sim \text{Bi}(n, p)$ where $p = \frac{\lambda}{\lambda + \theta}$.

   One way to test $H_0$ is to condition on the total number $n = (A + B)$ of accidents (which itself offers no evidence about $H_0$) and find a rejection region $R_n$ with the property that $P[A \in R_n : A + B = n] \leq \alpha$; the unconditional bound $P[A \in R_{A+B}] \leq \alpha$ will then follow. Carry out the test for $(A = 5, B = 2)$ and for $(A = 3, B = 0)$. You may find your solution to Problem 2 useful.

6. Find the posterior probability of $H_0$ in Problem 5 if $\lambda$ and $\theta$ have independent exponential prior distributions with mean one, for the same two observations as before.