For all three problems, let \( \{X_i : 1 \leq i \leq n\} \overset{iid}{\sim} \text{Ex}(\theta) \).

**Problem 1:** Consider the two possible hypotheses

\[ H_0 : \theta = \theta_0, \quad H_1 : \theta = \theta_1. \]

Find the posterior probability of \( H_0 \) if the prior probabilities are \( P[\theta = \theta_0] = p \in (0, 1) \), \( P[\theta = \theta_1] = q = (1 - p) \).

**Problem 2:** Now consider the hypotheses

\[ H_0 : \theta = \theta_0, \quad H_1 : \theta \neq \theta_0 \]

with prior distribution depending on three parameters \( p \in (0, 1), \alpha > 0, \beta > 0 \) given by

\[ P[\theta \in A] = p 1_A(\theta_0) + q \int_A \theta^{\alpha-1} e^{-\beta \theta} / \Gamma(\alpha) \, d\theta, \quad A \subset \mathbb{R}_+. \]

This is a mixture model, with a point mass of size \( p \) at \( \theta_0 \) and the rest of the prior mass \( q = (1 - p) \) distributed as \( \text{Ga}(\alpha, \beta) \).

a) Find the posterior odds \( P[H_0 \mid X] / P[H_1 \mid X] \). Simplify as much as possible.

b) The Jeffreys Prior distribution for the exponential distribution is \( \pi_J(\theta) \propto \theta^{-1} \), the limiting case of \( \text{Ga}(\alpha, \beta) \) as \( \alpha \to 0 \) and \( \beta \to 0 \). Find the limiting posterior odds.

**Problem 3:** Consider a sequential test of the hypotheses

\[ H_0 : \theta = \theta_0, \quad H_1 : \theta = \theta_1 \]

for the specific values \( \theta_0 = 2 \) and \( \theta_1 = 3 \).

a) Find the lower and upper limits \( a \) and \( b \) for \( \Lambda_n \) for a SLRT with approximate error probabilities \( \alpha \approx 0.01 \) and \( \beta \approx 0.02 \).

b) Describe the test precisely in terms of \( \bar{X}_n \) _i.e._, find the numbers \( L_n \) and \( R_n \) such that the test ends when \( \bar{X}_n \leq L_n \) or \( \bar{X}_n \geq R_n \), and in each case say whether to Reject \( H_0 \) or not. Simplify and be explicit.

c) Find the approximate expected sample size under both \( H_0 \) and \( H_1 \). Give answer numerically (but also show your work so we’ll know how you got it).

d) This SPRT may also be viewed as a sequential test of \( H_0 \) against the one-sided composite test \( H'_1 : \theta > \theta_0 \). Find its approximate power if in fact \( \theta = 2.75 \) _i.e._, find the probability that \( H_0 \) will be rejected. Hint: Find \( p \in \mathbb{R} \) s.t. \( \Lambda_n^p \) is a martingale if \( \theta = 2.75 \). Also find the approximate expected sample size needed if \( \theta = 2.75 \).