Sta 532: Homework #5

1. Fix $\alpha \in (-1, 1)$ and set $l(z) := |z| - \alpha z$ for $z \in \mathbb{R}$. For data $X \sim \mathsf{No}(\theta, 1)$ and improper uniform prior $\pi(\theta) \equiv 1$, verify the claim made in class notes that the statistic $T(X)$ that minimizes
$$r(\pi; T) = \mathsf{E}\Big[l\big(T(X) - \theta\big) \mid X\Big]$$
is the $(1+\alpha)/2$th quantile of the posterior distribution of θ— in particular, for $\alpha = 0$, the statistic that minimizes $\mathsf{E}\big[|T(X) - \theta| \,\big|X\big]$ is the median.[1]

2. For $\omega \sim \mathsf{Un}(0, \pi/2)$ set $p := \sin^2(\omega)$. Find the distribution of p.

3. For ψ with the improper uniform distribution $\pi(\psi) \equiv 1$ on the real line $\mathbb{R}$, find the (induced) distribution of the inverse logistic transform $p := 1/[1 + \exp(-\psi)]$.

4. Let $\mathfrak{F} = \{f(x \mid \theta) : \ \theta \in \Theta \subset \mathbb{R}\}$ be a one-dimensional parametric family of distributions with twice-continuously-differentiable pdf $f(x \mid \theta)$ on an interval $\Theta \subset \mathbb{R}$ and Fisher information $I^\theta(\theta) = \mathsf{E}_\theta\big[-(\partial^2/\partial\theta^2)\log f(X \mid \theta)\big]$. For a smooth 1:1 parameter transformation to $\phi = H(\theta)$, let $g(x \mid \phi) := f(x \mid \theta)$ be the pdf for X in the ϕ-parametrization. Find the Fisher Information $I^\phi(\phi) = \mathsf{E}_\phi\big[-(\partial^2/\partial\phi^2)\log g(X \mid \phi)\big]$ in terms of $I^\theta(\theta)$.

5. Again let $\phi = H(\theta)$ be a smooth 1:1 transformation from an interval space $\Theta \subset \mathbb{R}$, and let $\pi(\theta)$ be a pdf on Θ. Find the induced pdf $\xi(\phi)$ for $\phi = H(\theta)$, and verify that if $\pi(\theta) \propto \big[I^\theta(\theta)\big]^{1/2}$ then $\xi(\phi) \propto \big[I^\phi(\phi)\big]^{1/2}$. It is in this sense that Jeffreys' Rule prior is invariant to parameter changes.

6. Let $X, Y \overset{\text{iid}}{\sim} \mathsf{Ca}(\theta, 1)$ be independent Cauchy random variables with median θ and unit scale. Show that the likelihood function is multi-modal if and only if $|X - Y| > 2$. Suggestion: Without loss of generality you may take X and Y to be $\pm z$ for $z \geq 0$ (why?); what does the log likelihood function look like near $\theta = 0$?

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[1]Hint: Once X is observed, what is the (posterior) distribution for θ? As a function of $t = T(X)$, what is $\mathsf{E}[l(t - \theta) \mid X]$? How can you minimize this over all $t \in \mathbb{R}$?