Sta 532: Homework #6

Several of these problems involve the following random sample from some probability distribution:

$$\{X_i\} = \{1, 16, 13, 9, 30, 6, 2, 21, 1\} \qquad (\star)$$

1. Let $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Un}(0, \theta)$ with $\theta > 0$ unknown. Based on the sample $(\star)$ find a central 90% Confidence Interval for θ. Suggestion: First, find a pivotal quantity $T(x)$ based on a sufficient statistic.

2. Let $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Un}(0, \theta)$ with $\theta > 0$ unknown. Using an improper uniform prior density $\pi(\theta) = \mathbf{1}_{\{\theta > 0\}}$, find a central 90% Credible Interval for θ for sample $(\star)$.

3. Let $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Po}(\theta)$ be Poisson variables with mean $\theta > 0$. Find the Jeffreys' Rule prior density $\pi_J(\theta)$ for θ and, based on the sample $(\star)$, find a central 90% credible interval for θ.

4. For normal data $\{Y_i\} \overset{\text{iid}}{\sim} \mathsf{No}(\mu, 1)$ with unit variance, how large a sample size n is required for a 99% Confidence Interval $[L(\mathbf{y}), R(\mathbf{y})]$ for μ to have length $|R(\mathbf{y}) - L(\mathbf{y})|$ less than 0.01?

5. Let $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{No}(\mu, \sigma^2)$ be normally-distributed with uncertain mean and variance. Based on the sample $(\star)$ find a central 90% Confidence Interval for σ.

6. Find the Jeffreys' prior $\pi_J(\theta)$ for the Geometric distribution with pmf

$$\mathsf{P}_\theta[Y = y] = \theta(1-\theta)^y, \; y \in \mathbb{Z}_+ = \{0, 1, 2, ...\}$$

for some $\theta \in \Theta = (0, 1)$, by giving its name and the value(s) of any parameter(s). Also find the posterior distribution $\pi_J(\theta \mid \mathbf{y})$ for a sample $\mathbf{y} = \{Y_1, \cdots, Y_n\}$, the posterior mean $\mathsf{E}_J[\theta \mid \mathbf{y}]$, and the MLE $\hat{\theta}(\mathbf{y})$.

7. In fact the data in $(\star)$ were generated from a geometric distribution. Find an objective Bayes 90% Credible Interval for θ from these data under the model $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Ge}(\theta)$.

8. Find a central 90% Confidence Interval for θ based on the sample ($\star$), for the model $\{X_i\} \overset{\text{iid}}{\sim} \mathsf{Ge}(\theta)$.

   Suggestion: You will need to know (or figure out) the probability distribution for the natural sufficient statistic $T(\mathbf{x})$ for a sample of size n from $\mathsf{Ge}(\theta)$, and recall from course notes "ci.pdf" that for integer-valued distributions with CDF $F_\theta(x) = \mathsf{P}_\theta[X \le x]$, a central $100\gamma\%$ confidence interval $[L(X), R(X)]$ can be found by evaluating

   - If $F_\theta(x)$ is monotone **decreasing** in θ for each fixed x,

     $$L(x) := \sup\left\{\theta :\ F_\theta(x-1) \ge \tfrac{1+\gamma}{2}\right\} \quad R(x) := \inf\left\{\theta :\ F_\theta(x) \le \tfrac{1-\gamma}{2}\right\}$$

   - If $F_\theta(x)$ is monotone **increasing** in θ for each fixed x,

     $$L(x) := \inf\left\{\theta :\ F_\theta(x) \le \tfrac{1-\gamma}{2}\right\} \quad R(x) := \sup\left\{\theta :\ F_\theta(x-1) \ge \tfrac{1+\gamma}{2}\right\}$$

   any of which can be found using `R`.

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