Sta 961: Homework #3

1. Verify that the Markov change-point gamma process does *not* have ID marginal distributions of all orders— in fact, even the two dimensional marginal distributions (whose joint chf is given in §2.4 of the lecture notes) cannot be written as the sum of $n = 2$ iid random variables.

2. Fix $\lambda > 0$ and $0 < \rho < 1$. If $X \sim \mathsf{Po}(\lambda)$ and, conditional on X, $Y \sim \mathsf{Bi}(X, \rho)$, find the joint distribution of Y and $Z := X - Y$, $\mathsf{P}[Y = y,\ Z = z]$.

3. Fix $\lambda > 0$ and $0 < \rho < 1$. Construct a stationary Markov chain $\{X_t\}$ with marginal distribution $X_t \sim \mathsf{Po}(\lambda)$ and autocorrelation $\mathrm{Corr}(X_s, X_t) = \rho^{|s-t|}$ for which the pair (X_0, X_1) has a bivariate ID distribution.

4. Write a program in your choice of R, Matlab, or Python that draws $\{X_t\}$ at integer times $0 \le t \le 100$ for one or more of the six stationary AR(1)-like Gamma processes described in the class notes, with $\mathsf{Ga}(\alpha = 4, \beta = 1)$ marginal distributions and with one-step autocorrelation $\rho = \exp(-\lambda) = 0.90$. For each,

   - plot X_t vs. t;
   - show a histogram of $\{X_t\}$ with the $\mathsf{Ga}(\alpha, \beta)$ pdf overlaid;
   - show a scatter-plot of consecutive pairs $\{(X_t, X_{t+1})\}$;
   - compute the sample mean, variance, and one-step autocorrelation, and compare them to the theoretical values;
   - comment on any features you notice that distinguish the processes.

   Generating the Gamma $\mathsf{AR}(1)$ process of §2.1 and the Thinned Gamma Process of §2.2 recursively (first X_0, then X_1, etc.) is straightforward. Generating the Squared O-U Gamma Diffusion at integer times is straightforward too, since the OU processes $Z_i(t)$ at integer times are just Gaussian $\mathsf{AR}(1)$ processes with $\exp(-\lambda/2) = \sqrt{\rho}$ for the autocorrelation parameter, so they too can be constructed recursively.

   The Markov change-point Gaussian Process can constructed at integer times either recursively or, a little more elegently, by noticing that there will be a $\mathsf{Po}(100\lambda)$-distributed number N of change-points in the entire 100-unit period, whose distribution, conditionally on N, will be iid $\mathsf{Un}(0, 100)$.

   The Continuously-Thinned Gamma process can be approximated by either drawing a few thousand mass points $\{(b_j, t_j)\}$ and $\{(\zeta_i, s_i)\}$ of the Beta and Gamma processes and constructing explicitly the stochastic integral (equation (12b) in §2.6 of the class notes), or simply by choosing a large integer n and constructing the sum of equation (12a) as an approximation.

Finally, there remains the Random Measure construction of §2.3. For this set $T = 100$, $\rho = 0.9$, and construct the process X_t at times $t \in \{0, \ldots, T\}$ as finite sums (see Figure (1))

$$X_t = \sum_{i=0}^{t} \sum_{j=t-i}^{T-i} \zeta_{ij}, \qquad 0 \le t \le T \tag{1}$$

of independent random variables

$$\zeta_{ij} \sim \mathsf{Ga}(\theta_{ij}\alpha, \beta)$$

with intensities θ_{ij}, for $0 \le i \le i + j \le T$, given by

$$\theta_{ij} = \begin{cases} \rho^j (1-\rho)^2 & 0 < i \le i+j < T; \\ \rho^j (1-\rho) & 0 < i \le i+j = T; \\ \rho^j (1-\rho) & 0 = i \le i+j < T; \\ \rho^j & 0 = i \le i+j = T. \end{cases}$$

This entails drawing $T(T+1)/2 = 5151$ gamma-distributed ζ_{ij}s. As an alternative, es-

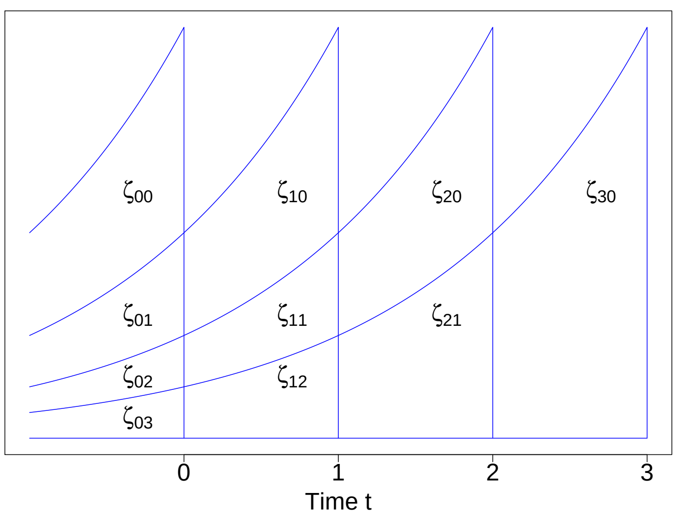

Figure 1: Illustration of random-measure $\mathsf{AR}(1)$-like Gamma process X_t

pecially appealing if X_t is required at a larger number T of times (or continuously at all t in some interval), one can draw the largest $1,000$–$10,000$ or so mass points $\{(x_i, y_i)\}$ of the random measure $\mathcal{G}(dx\,dy)$ using the Inverse Lévy Measure (ILM) algorithm of Wolpert & Ickstadt (1998), and evaluate $X_t = \mathcal{G}(G_t)$ where needed.

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